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Chemistry. — *On colloidally bound water.* By H. R. KRUYT and H. DE BRUYN Mzn.

(Communicated at the meeting of May 25, 1940.)

I. *Introduction.*

The determination of colloidally bound water has been tried along various ways and the results of these investigations have led to contradictory conclusions.

Leaving the viscosimetric investigations apart (on account of the difficulty they present of correct interpretation), there remain four types of investigation:

1. Determination of water which does not freeze into ice at low temperatures. As a criterion we can take the measurement of the melting heat at a rise of temperature (RUBNER 1), THOENES 2)), or the volume change (JONES and GORTNER 3)).

2. Determination of the actual concentration of a substance added to the sol; the difference between the total quantity of water and that which proves available as solvent, is the colloidally bound water. The concentration may be determined by cryoscopic measurement (NEWTON and GORTNER 4)), or polarimetrically or volumetrically (KOETS 5) in those cases when the liquid can be simply poured away from the colloid.

3. Separation of colloid and medium may also be done by means of ultrafiltration (GREENBERG and GREENBERG 6)). The criterion is, of course, the difference between the total quantity of water and that which serves as a colvent for a molecular dispersion substance (concentration of the ultrafiltrate).

4. Determination of the difference in concentration of a dissolved substance on either side of a dialysis membrane, when on the one side there is colloid, water and dissolved substance, on the other only water and dissolved substance (OAKLEY 7)); the interpretation is again as in 2 and 3.

The remarkable fact is, that methods 1, 2 and 4 have led to similar results, at least as regards the order of magnitude (water binding in the order of 1 gr of water per gram of colloid), whereas method 3 gives an absolutely negative result. This is especially striking in methods 3 (GREENBERG) and 4 (OAKLEY), which differ only in that the one is by ultrafiltration, the other by dialysis.

The calculation of the results of all these methods easily causes errors; the concentrations should be calculated per water unit, not per solution unit. Some communications in the literature are so concise, that it is impossible to check the method of calculation. We have, therefore, thought it necessary to investigate the same objects by different methods and by calculations which are in every respect comparable. We have, however, abstained from checking method 1, because we deem its results impossible of interpretation without suppositions which cannot be verified themselves; method 2 we have left unchecked, because two widely different investigations in this laboratory (KOETS 5), KRUYT and WINKLER 8)) have already confirmed the correctness of the conclusions drawn from this method.

We have, therefore, occupied ourselves with OAKLEY's and GREENBERG's methods, which, for all their similarity, have led to such opposite conclusions. As objects of both methods we took: *sodiumarabinate* for colloid and urea for the moleculary dispersed substance, whose concentration was to be determined. We have applied OAKLEY's and GREENBERG's methods, we have modified them, made supplementary investigations where it seemed necessary, yet *we have had to fully confirm the apparently paradoxical result*: "in OAKLEY's method there is bound water, in GREENBERG's there seems to be none".

In the following experimental part a complete description is given of some of the measurements taken, so that others may check our results and get an idea of the limits of accuracy. In the treatises published on this subject this is often impossible, it has been our wish to prevent this uncertainty in the reader.

II. *Experimental part.*

A. OAKLEY's dialysis method.

1. *Material.*

The *Na arabinate* was prepared by the method of BUNGENBERG DE JONG and THEUNISSEN 9); starting from gum Arabic "Senegal, petit boule blanche I", it was precipitated four times with acetate of sodium and alcohol.

The urea was a *purissimum* preparation, the purity of it was checked by analysis, as described below.

2. *Analysis.*

The urea concentration was determined by the micro-KJELDAHL method namely, following HENWOOD and CAREY 10). This method was first tested by numerous blank determinations, with known quantities of urea.

3. *Investigation*

We made little collodion bags by fabricating four successive collodion films, one on top of the other in ERLEMEYER flasks of 25 cc. These bags were pushed over a length of vacuum tube, in which a glass tube had been introduced. With this glass tube the dialysis membrane was secured and placed in a covered beaker containing the sol-free solution, which was renewed after 1, 2, 4, 6 and 10 days respectively; after a fortnight 10 cc of the inside and 10 cc of the outside liquid were analyzed; moreover, 5—10 cc of the inside liquid were dried in a vacuum drier over P_2O_5, in order to determine the quantities of gum arabic and urea.

Only in experiment 5 did we analyze after a month, after the outside liquid had been renewed 10 times. From the figures obtained in this way we calculated "the quantity of bound water per gram of colloid, *h*", according to the formula given by GREENBERG:

$$h = \frac{1}{c}\left(1 - \frac{x}{y}\right),$$

in which x is the urea concentration per gram of the total quantity of water in the colloid solution inside the dialysator, y the same concentration but in the beaker, so outside the dialysator, c the colloid concentration in grams per gram of water.

Table I shows that the quantity of bound water is about 1 gram per gram of *sodiumarabinate* and that in an *arabinate* solution of ca 5 %. In 3—8 % concentrations OAKLEY found 1.1 gram.

In principle, therefore, our results confirm those of OAKLEY.

Meanwhile the objection may be raised that possibly in OAKLEY's method the equilibrium has not yet been established after a fortnight and 5 renewals. Although experiment 5, in which, as has been remarked, 10 renewals were made in 30 days, does not point in that direction, we thought it would be well to approach the equilibrium from the other side, namely in two experiments (I and II) we placed a greater urea concentration inside the dialysator than outside, while afterwards in experiment III the concentrations were chosen so that acocrding to experiments I and II there was equilibrium in concentration from the first. In order to promote the establishment of equilibrium this series of experiments was made with somewhat thinner collodion membranes, but owing to this a slight quantity of gum arabic permeated, so that this substance had also to be determined in the outside liquid (Table II).

In this series intermediary determinations were also made. We shall not describe the results extensively, but restrict ourselves to publishing the figures (Table III).

The results of these experiments do not leave any doubt that there is bound water, namely, one to one and a half grams per gram of arabinate.

B. GREENBERG's ultrafiltration method.

1. The material used and the method of analysis were identical to those applied with OAKLEY's method.

TABLE I.

	1		2		3		4		5	
	outside	inside	outside	inside	outside	inside	outside	inside	outside	inside
g. liquid	10.0800	9.9922	10.0082	10.0480	9.9984	10.0677	10.0604	10.0437	10.0863	10.0265
g. urea	0.1210	0.1089	0.1202	0.1071	0.1196	0.1060	0.1202	0.1076	0.1209	0.1073
idem/g. liquid	0.01201	0.01090	0.01201	0.01066	0.01196	0.01053	0.01195	0.01071	0.01199	0.01070
g. liquid		5.1171		10.1044		9.9864		10.0888		8 6494
g. dry substance		0.2996		0.6585		0.6561		0.6516		0.5467
g. gum arabic		0.2438		0.5508		0.5510		0.5435		0.4541
idem/ liquid		0.04764		0.05451		0.05517		0.05387		0.05250
water/liquid	0.9880	0.9415	0.9880	0.9348	0.9880	0 9343	0.9880	0.9354	0.9880	0.9368
h.	0.93		1.06		1.17		0.92		1.05	

TABLE II.

	I		II		III	
	outside	inside	outside	inside	outside	inside
g. liquid		20.5969		20.5167		20.5559
g. urea		0.2424		0.2555		0.2158
id/g liquid	0.01120	0.01177	0.01120	0.01245	0.01197	0.01050
g. gum/g liquid		0.09882		0.09876		0.09895
water/g liquid	0.9888	0.8894	0.9888	0.8888	0.9880	0.8905
after days	12		13		10	
g. liquid	10.0295	9.7520	10.0430	10.0613	10.0390	9.5945
g. urea	0.1137	0.1064	0.1163	0.1114	0.1198	0.1081
id/g. liquid	0.01134	0 01091	0.01158	0.01107	0.01193	0.01127
g. liquid	6.0074	5.5306	6.5000	2.9325	5.3543	3.7023
g. dry subst.	0.1289	0.1937	0.1043	0.0957	0.0722	0.1307
g. gum	0.0608	0.1334	0.0290	0.0632	0.0083	0.0809
id/g. liquid	0.01012	0.02412	0.00446	0.02155	0.00155	0.02404
water/g. liquid	0.9785	0.9650	0.9840	0.9674	0.9865	0.9647
h	1.69		1.56		1.46	

TABLE III.

	After days	h
I	6	—
	12	1.69
II	11	1.03
	13	1.56
III	3	0.78
	8	1.02
	10	1.46

2. *Investigation.*

The ultrafiltration took place at diminished pressure; seeing that in such a case there is a danger of loss of water of the filtrate through evaporation, two calcium chloride tubes were placed behind the suction tube. We found that the weight of the second tube did not increase, while the water received in the first was taken into account.

The first experiments were made with "Ultrafein-Schnell" membrane filters and yielded the results stated in Table IV.

TABLE IV.

	I		II	
	Filtrate	Liquid	Filtrate	Liquid
g. liquid	7.9816		10.0017	
g. urea	0.0964		0.1317	
idem/5. liquid	0.01208	0.01088	0.01317	0.01210
g. gum/g liquid		0.09091		0.09880
water/g liquid	0.9879	0.8982	0.9868	0.8891
h	0.09		— 0.18	

These results, in which *h* is calculated in the same way as in the investigations according to Oakley's dialysis method, show that indeed, the bound water thus calculated is practically nil in Greenberg's ultrafiltration method.

In order to make the two methods as much alike — and hence as comparable — as possible, we applied in the ultrafiltration the same collodion membrane as in the Oakley experiments. It was simply placed in a porous beaker (ultrafilter of Bechhold–König 11)). This gave a double advantage: in the first place the material was the same in both series of experiments, secondly the collodion membrane is much more permeable, so that it is possible to filter at less diminished pressure. For the dialysis method may be conceived as ultrafiltration in which the difference in pressure is nil. In order to achieve different diminished pressures with the same water-jet pump, an adjustable regulator was placed behind the suctionflask and the working pressure was read from an open mercury manometer. It has been said that the thin collodion membranes allow a slight quantity of arabinate to pass, this was determined in the filtrates. The level of filter beaker was kept constant by adding sol and the homogeneousness of the liquid was ascertained by stirring. The results are given in Table V.

These experiments too confirm, therefore, Greenberg's result that ultra-

TABLE V.

	I				II			
g. urea/g. liquid	original liquid			0.01192				0.01209
g. gum/g. liquid				0.02351				0.02287
g. water/g liquid				0.9646				0.9650
	filtrate	liquid	filtrate	liquid	filtrate	liquid	filtrate	liquid
g. liquid	8.3615		8.6531	8.2633	8.3034		7.5265	8.2450
g. urea	0.1032		0.1062	0.0988	0.1032		0.0949	0.1017
g. urea/g. liquid	0.01234	0.01216	0.01228	0.01196	0.01243	0.01230	0.01261	0.01233
g. liquid	7.3431		7.0880	7.4985	3.6588		3.8708	5.7796
g dry substance	0.0905		0.0915	0.4080	0.0556		0.0559	0.2662
g. gum	—		0.0045	0.3183	0·0101		0.0071	0.1949
idem/g. liquid	—	0.0333	0.00063	0.04245	0.00276	0.0303	0.00183	0.03372
water/g. liquid	0.9877	0.9545	0.9871	0.9456	0.9848	0.9574	0.9856	0.9540
pressure in cm Hg	60		60		20		31	
h	— 0.56		— 0·38		— 0.62		— 0.30	

filtration experiments do not lead to the conclusion that bound water is real, rather do they point to a negative effect.

III. *Discussion.*

Although we can fully confirm the experimental results of the authors mentioned, we yet think we must contradict the conclusion that GREENBERG's method indicates the absence of colloidally bound water.

As well in dialysis as in the ultrafiltration method (at least with infinitely slow ultrafiltration and uninterrupted contact of the two liquids) the potential of the water on either side of the membrane should be equal. The question whether or not the water is bound may be formulated as follows: is the potential of the water modified by the colloid or not? If we give an affirmative answer to this question on the ground of GORTNER's and OAKLEY's experiments, we may express the potential of the water inside the ultrafilter, the dialysis membrane respectively, as

$$[\mu_{H_2O}]_{in} = f(T) + p_{in}\, v_0 + f(c_f)_{in} + f(c_c)_{in} \quad . \quad . \quad . \quad . \quad (1)$$

in which c_t is the concentration of the substance in true solution, c_c that of the colloid, and v_0 the molecular volume of H_2O. Outside the ultrafilter, the dialysis membrane respectively, applies:

$$[\mu_{H_2O}]_{out} = f(T) + p_{out}\, v_0 + f(c_t)_{out} \quad . \; . \; . \; . \; . \quad (2)$$

To hold for equation:

$$[\mu_{H_2O}]_{in} = [\mu_{H_2O}]_{out} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

term $f(c_c)_{in}$, in equation (1) will have to be compensated either by a difference between p_{in} and p_{out}, or by one between $(c_t)_{in}$ and $(c_t)_{out}$. In OAKLEY's method, where the pressure is the same inside and outside the membrane, there is indeed a concentration difference as equilibrium phenomenon. In GREENBERG's method there is a difference in pressure between sol and ultrafiltrate; hence it is by no means certain that a concentration difference is necessary, as long as $p_{in} - p_{out}$ is sufficiently great.

In calculating what difference in pressure agrees with the concentration difference found, we find a value of ca. $\frac{1}{4}$ atmosphere; in our experiments we were always higher than that. Moreover, it should be borne in mind that in the ordinary ultrafiltration the condition of infinitely slow ultrafiltration with permanent maintenance of the equilibrium is by no means fulfilled, owing to which an inversion of the phenomenon for higher pressure values becomes doubtful. The homogeneous liquid phase simply permeates, and as it becomes separated from the membrane, equation (3) is not held for.

GREENBERG's *method, therefore, by no means contradicts the presence of bound water,* OAKLEY's *method confirms the existence of it unequivocally.*

Utrecht, April 1940.

VAN 'T HOFF *Laboratorium of the Rijks Universiteit.*

REFERENCES.

1) M. RUBNER: Abh. preuss. Akad. Wiss., Phys.-Math. Kl. **1922**, 3.
2) F. THOENES: Biochem. Z. **157**, 174 (1925).
3) I. D. JONES and R. A. GORTNER: J. physic. Chem. **36**, 387 (1935).
4) R. NEWTON and R. A. GORTNER: Botan. Gaz. **74**, 442 (1922).
5) P. KOETS: Proc. Kon. Akad. v. Wetensch., Amsterdam, **34**, 420 (1931).
6) D. M. GREENBERG and M. GREENBERG: J. Biol. Chem. **94**, 373 (1931).
D. GREENBERG and W. C. KOHN: J. Gen. Physiol. **16**, 559 (1932—'33); **18**, 93 (1935).
7) H. B. OAKLEY: Biochem. J. **31**, 28 (1937).
8) H. R. KRUYT and K. C. WINKLER: Z. anorg. Ch. **188**, 200 (1930).
9) H. G. BUNGENBERG DE JONG and P. H. THEUNISSEN: Koll. Beih. **47**, 254 (1938).
10) A. HENWOOD and R. M. GAREY: J. Frankl. Inst. **221**, 531 (1936).
11) H. BECHHOLD—A. KÖNIG: D. R. P. 403405 Kl. 421 (1924).

Physics. — *Decrease in the intensity of cosmic rays in different directions and the decay of the mesons.* II. By J. CLAY.

(Communicated at the meeting of May 25, 1940.)

In a previous communication ¹), we have given a comparison of the decrease of cosmic rays in different directions by the atmosphere and the decrease by a layer of water in a vertical direction. As we had in the vertical direction no measurements from 0—10 m water, it was impossible to compare the influence of smaller layers with the decrease in the atmosphere for small angles with the vertical.

It was possible to make a comparison of the decrease for layers of lead in the vertical direction and the decrease in the atmosphere for small angles. And only for small angles it will be possible to distinguish between the two possibilities, if either the decay of the mesons or the abnormal absorption of the gaseous state of matter is the reason of the abnormal behaviour of the rays in the atmosphere, as was supposed by FERMI ²).

We counted threefold coincidences with counterboxes of 846 cm² active surface. The distance between the upper and the lower counter was 100 cm (fig. 1). The boxes with counters and lead were fixed on an iron plate of 2 m length and we could change the inclination of the plate.

Between the lower and the middle counter we had a layer of 30 cm Pb, so that we could be sure that only particles of high penetration power were counted. The number of coincidences in vertical direction was 25.8 per min. The relation of the intensity with inclination can be presented very well by $\cos^2 \varphi$ as can be controled by comparing column 3 and 5 and in fig. 2. In column 7 (Table I) the layer of water is given aequivalent to the layer of lead in 6. 1 cm Pb is aequivalent to 6,9 cm H_2O ³).

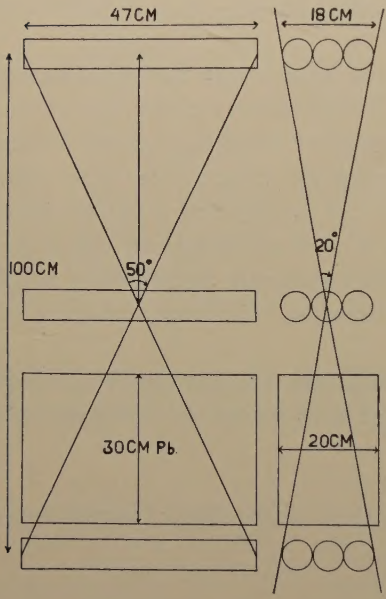

Fig. 1. Countersystem used for measurements in vertical and oblique directions. 30 cm Pb between the counters.

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¹) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 436 (1940).

²) E. FERMI, Phys. Rev. **56**, 1242 (1939).

³) J. CLAY and A. v. GEMERT, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 672 (1939).

1	2	3	4	5	6	7	8
φ	sec φ	$\cos^2 \varphi$	a m water-aeq.	I_φ	b' cm Pb for $I = I_\varphi$	b M water for $I = I_\varphi$	$\frac{b}{a-10}$
0	1.00	1.00	0	1.000	0	0	
10	1.015	0.970	10.15	0.968	6	0.41	2.7
15	1.034	0.932	10.34	0.950	15	1.03	3.05
20	1.060	0.881	10.60	0.875	24	1.67	2.80
30	1.155	0.751	11.55	0.755	53	3.66	2.36
40	1.306	0.586	13.06	0.604	90	6.20	2.02
45	1.414	0.500	14.1	0.504	122	8.41	2.05
45	1.414	0.500	14.1	0.484		8.3	2.0
60	0.500	0.250	20.0	0.221		20	2.0
75	3.89	0.067	38.9	0.072		47	1.7
82	7.20	0.020	72	0.019		110	1.8

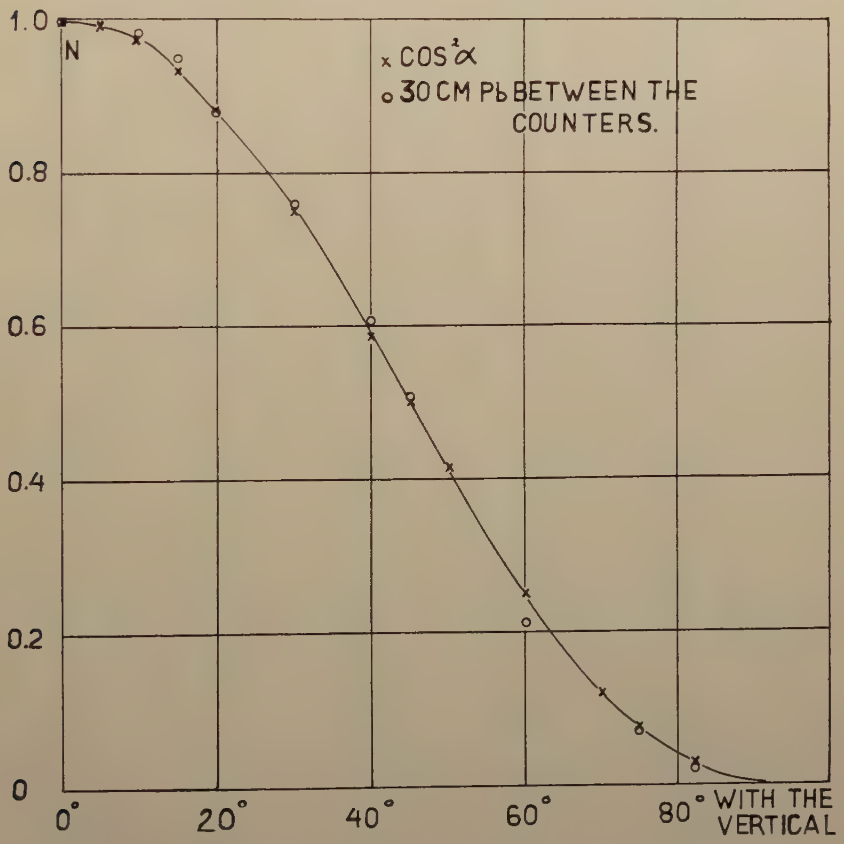

Fig. 2. Variation of the intensity in different directions.

For the series of angles between 45° and 82° only the layer of water could be given, found from the absorption curve in water between 10 and 200 m of depth [1]).

In column 8 the relation is given between the layer of water, transversed in vertical direction and the wateraequivalent of the atmospheric layer, which both give the same decrease of intensity. The relation of these values is also given in fig. 3 for the whole series of inclinations and in fig. 4

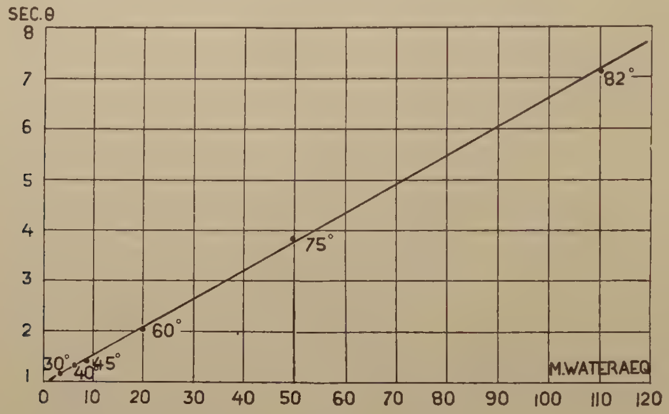

Fig. 3. Relation between sec θ and thickness of waterlayer (b) for aequivalent decrease 20°—82°.

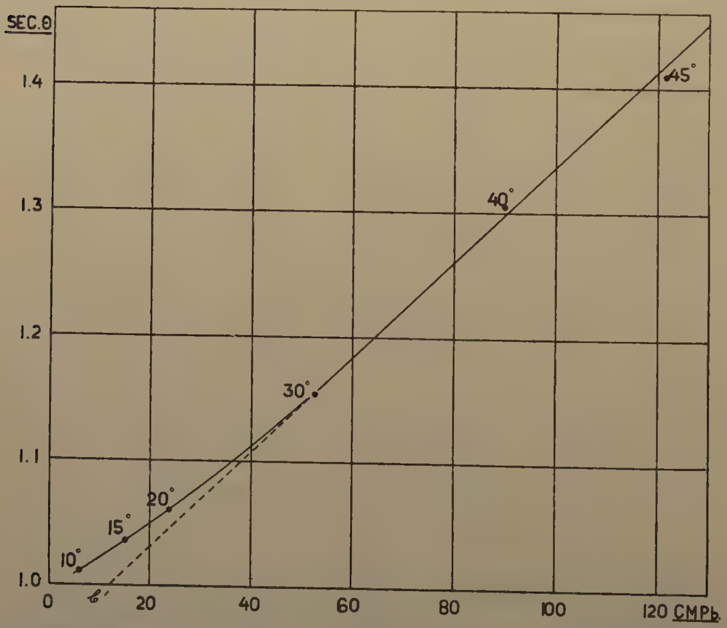

Fig. 4. Relation between sec θ and leadthickness for small angles with the vertical.

for the inclinations to 45° and the accurate values of lead. The decrease in vertical direction can be found in fig. 5.

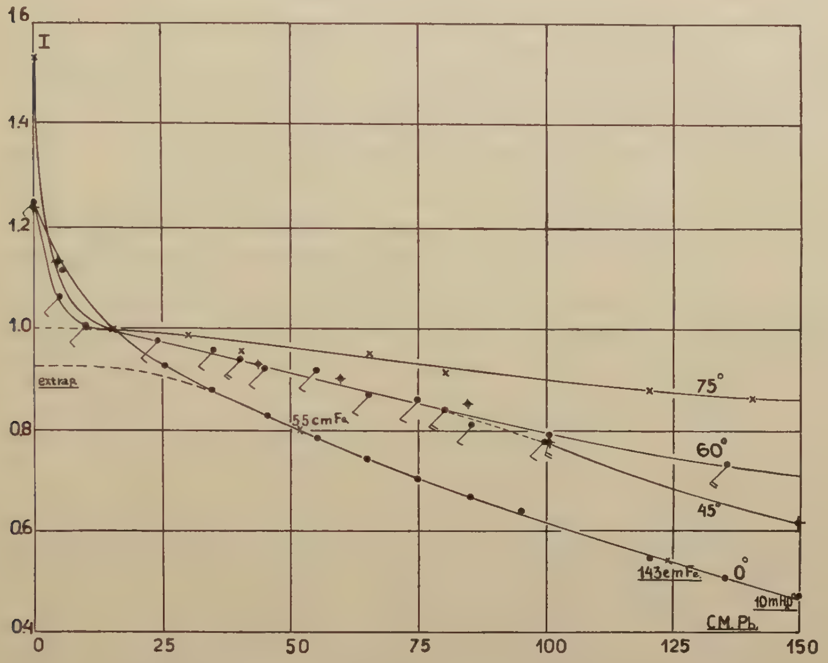

Fig. 5. Decrease of intensity in lead in vertical direction.

If the reason of the abnormal decrease of the atmosphere was the distance of the atoms in the gaseous state, as was supposed by FERMI, there had to be a b (and b′) proportional to sec θ down to $\theta = 0$.

We see from the graph that this is not according to experiment. If the decay of mesons is the origin of the abnormal high decrease in the atmosphere the relation sec θ to b will be linear for higher values of θ as well, but will be changed for the lower values as is expected from the calculations of BRUINS [4]).

We see that the experimental results are in agreement with this second supposition.

The value of the inclination of the line $\dfrac{d(\sec\theta)}{db'} \times 146 = 0{,}61$, means that the lifetime of the particles is $2{,}6 \times 10^{-6}$ sec. vid BRUINS [4]) graph. 2. As db' is given in cm Pb we have to multiply by 146 as 10 m water is aequivalent to 146 cm Pb.

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[4]) E. M. BRUINS, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, p. 700 (1940).

Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im R_4.* (Dritte Mitteilung). Von R. WEITZENBÖCK.

(Communicated at the meeting of May 25, 1940.)

Ich setze im Anschlusse an die vorige Mitteilung die Ermittlung der zu einer allgemeinen Erzeugenden der Fläche F gehörigen linearen Räume fort, wobei die Voraussetzung gemacht wird, dass die (einfachste) Differentialinvariante Q nicht verschwindet. Als besondere Tatsache sei hervorgehoben, dass die Heftebene von der zur Erzeugenden „assoziierten" oder Bei-Geraden getroffen wird.

§ 9.

Die durch (32) gegebene Heftebene m_{ijk} der Erzeugenden 0_{ik} kann nach Gleichung (35) auch in der Gestalt

$$m\,(0_{ik}) = (1^2 H \pi^2) = 0$$

dargestellt werden. Wir ersetzen nun 0_{ik} nach (60) durch

$$\varphi_{ik} = \tfrac{3}{2}\,(HH_1)_{ik} + \lambda\,.\,0_{ik},$$

d.h. wir berechnen

$$m_{ijk}\,(\varphi_{ik}) = m\,(\varphi) = (\varphi_1{}^2\, H\,(\varphi)\,\pi^2).$$

Nach (62) und (70) wird

$$m\,(\varphi) = 3\,(HH_2H\,(\varphi)\,\pi^2) + \tfrac{3}{2}\,\lambda\,.\,(1^2\,H\,(\varphi)\,\pi^2) + \tfrac{3}{2}\,\lambda'\,(0^2\,H\,(\varphi)\,\pi^2) =$$
$$= \tfrac{3}{2}\,\lambda^2\,(\lambda - 8\,Q)^2\,.\,(1^2\,H\pi^2),$$

also

$$m_{ijk}\,(\varphi) = \tfrac{3}{2}\,\lambda^2\,(\lambda - 8\,Q)^2\,.\,m_{ijk}, \quad \ldots\ldots \quad (78)$$

wovon als besonderer Fall für $\lambda = 4\,Q$ folgt:

$$m_{ijk}\,(a) = 3.2^7\,.\,Q^4\,.\,m_{ijk}; \quad \ldots\ldots\ldots \quad (79)$$

weiters haben wir nach (59)

$$m_{ik} = a_{ik} + 4\,Q\,.\,0_{ik}.$$

Hieraus wird

$$m_{ik}\,(\varphi) = a_{ik}\,(\varphi) + 4\,Q\,(\varphi)\,.\,\varphi_{ik},$$

also nach (76) und (74)

$$m_{ik}(\varphi) = \lambda^3 (\lambda - 8\,Q)^3 \,.\, m_{ik}. \quad \ldots \ldots \quad (80)$$

Hieraus wieder für $\lambda = 4\,Q$:

$$m_{ik}(\alpha) = -(4\,Q)^6 \,.\, m_{ik}. \quad \ldots \ldots \quad (81)$$

Aus (80) folgt noch, dass der nach (42) zu berechnende Punkt M für die Gerade φ_{ik}, also der Punkt $M(\varphi)$, wieder der Geraden m_{ik} angehört.

Dem in (42) stehenden Ausdruck

$$R_0 = 2_{05,03}$$

lässt sich, wenn man die Ableitung Q' von $Q = 0_{13,23}$ benutzt, eine andere Gestalt geben. Wir haben

$$Q' = 1_{13,23} + 0_{23,23} + 0_{14,23} + 0_{13,33} + 0_{13,24}.$$

Hier verschwindet rechts der zweite Term, der vierte ist R (Vgl. (41)). Auch der dritte Term ist nach (1) Null. Beim ersten Term haben wir auf Grund der Gleichungen (1) und der zyklischen Symmetrie:

$$1_{13,23} = -2_{13,13} - 3_{13,12} = +\tfrac{1}{2}\,1_{33,12} = -\tfrac{1}{6}\,1_{33,03} = +\tfrac{1}{6}\,0_{33,13} = -\tfrac{1}{6}\,R;$$

und beim letzten Term wird analog

$$\begin{aligned} 0_{13,24} &= -\tfrac{2}{3}\,0_{13,33} - \tfrac{2}{5}\,0_{13,15} = -\tfrac{2}{3}\,R + \tfrac{2}{5}\,1_{03,15} = -\tfrac{2}{3}\,R + \tfrac{1}{5}\,5_{03,11} = \\ &= -\tfrac{2}{3}\,R + \tfrac{1}{5}\,5_{03,02} \quad = -\tfrac{2}{3}\,R + \tfrac{1}{5}\,2_{05,03} = -\tfrac{2}{3}\,R + \tfrac{1}{5}\,R_0. \end{aligned}$$

Also kommt schliesslich

$$R_0 = 2_{05,03} = 5\,Q' - \tfrac{5}{6}\,R \quad , \quad Q' = \tfrac{1}{6}\,R + \tfrac{1}{5}\,2_{05,03}. \quad \ldots \quad (82)$$

Diese Grösse R_0 tritt auch in der Gleichung der Schmiegebene h_{ijk}

$$(h'\pi)^2 = (\pi^2\,HH_1H_2) = 0 \quad , \quad h_{ijk} = (HH_1H_2)_{ijk}. \quad \ldots \quad (83)$$

im Punkte H an die Kurve C_H auf. Nach (21) haben wir

$$\begin{aligned} (\pi^2\,HH_1\,H_2) = (0H_1\,H_2\,\pi^2)\,0_{22} = \tfrac{1}{2}\,(0^2\,H_2\,\pi^2)\,(H_1)_{22} - \\ - \tfrac{1}{2}\,(0^2\,H_1\,\pi^2)\,(H_2)_{22} - (0^2\,H_1\,H_2\,\pi)\,\pi_{22}. \end{aligned}$$

Dies kann mit Hilfe von (54) und (55) umgeformt werden auf

$$(h'\pi)^2 = (\pi^2\,HH_1\,H_2) = \tfrac{4}{3}\,Q\,(\pi_{02,04} + \pi_{02,22}) + \tfrac{1}{3}\,\pi_{02,03}\,(H_2)_{22}. \quad . \quad (84)$$

Hier enthält der letzte Term die Heftebene m_{ijk} (Vgl. (32)) und für den Koeffizienten $(H_2)_{22}$ ergibt sich nach (55):

$$(H_2)_{22} = 4.1_{22,23} + 2.0_{22,33} + 2.0_{22,24}.$$

Nun ist wie oben:

$$1_{22,23} = -3_{22,12} = +\tfrac{1}{3}\,3_{22,03} = -\tfrac{1}{6}\,0_{22,33} = \tfrac{2}{9}\,R$$

$$0_{22,34} = -\tfrac{4}{3}\,R$$

$$0_{22,24} = -\tfrac{2}{3}\,0_{22,33} - \tfrac{2}{5}\,0_{22,15} = \tfrac{8}{9}\,R + \tfrac{2}{5}\,1_{22,05} = \tfrac{8}{9}\,R - \tfrac{4}{5}\,2_{12,05} =$$
$$= \tfrac{8}{9}\,R + \tfrac{4}{15}\,2_{03,05} = \tfrac{8}{9}\,R - \tfrac{4}{15}\,R_0.$$

Also wird

$$(H_2)_{22} = -\tfrac{8}{15}\,R_0 = \tfrac{4}{9}\,R - \tfrac{8}{3}\,Q'. \quad \ldots\ldots \quad (85)$$

Der Schmiegraum H' im Punkte H an die Kurve C_H wird gegeben durch die Kovariante

$$(H'x) = (HH_1\,H_2\,H_3\,x) = 0. \quad \ldots\ldots\ldots \quad (86)$$

Dies kann man am einfachsten nach (84) entwickeln indem man dort $\pi_{ik} = (H_3\,x)_{ik}$ einsetzt. Wir erhalten

$$(H'x) = \tfrac{4}{3}\,Q\,[x_{04}\,(H_3)_{02} - x_{02}\,(H_3)_{04} + x_{22}\,(H_3)_{02} - x_{02}\,(H_3)_{22}] +$$
$$+ \tfrac{1}{3}\,(H_2)_{22}\,[x_{03}\,(H_3)_{02} - x_{02}\,(H_3)_{03}]$$

$$\left.\begin{aligned}(H'x) = -x_{02}\,[\tfrac{4}{3}\,Q\,(H_3)_{04} + \tfrac{4}{3}\,Q\,(H_3)_{22} + \tfrac{1}{3}\,(H_2)_{22}\,(H_3)_{03}] +\\ + x_{03}\cdot\tfrac{8}{9}\,Q\,(H_2)_{22} + \tfrac{32}{9}\,Q^2\,[x_{04} + x_{22}]\end{aligned}\right\} \quad . \quad (87)$$

Hieraus ist schliesslich für $x = H_4$ ein Ausdruck für die Invariante

$$\triangle_H = (HH_1\,H_2\,H_3\,H_4) \quad \ldots\ldots\ldots\ldots \quad (88)$$

erhältlich, der $\triangle_H$ durch die Grössen $(H_i)_{rs}$ ausdrückt.

§ 10.

Wenn eine Regelfläche F abwickelbar ist, so verschwindet $(M'_{02}\,x)$ und daher auch (Hu') identisch. Ich behaupte, dass für diesen Fall die Differentialkomitante erster Ordnung

$$E_0 = 0_{1\pi}\,(0u') = (01^2\pi^2)\,(0u') = -(10^2\pi^2)\,(1u') \quad \ldots \quad (89)$$

den zugehörigen Kurvenpunkt (bei einem Kegel die Spitze) und die zugehörige Schmiegebene darstellt.

Beweis. Bei einer Torse haben wir $0_{ik} = (yy')_{ik}$, also

$$E_0 = (1\,yy'\,\pi^2)\,(1u') = -\tfrac{1}{2}\,(yy'\,y''\,\pi^2)\,.\,(yu');$$

bei einem Kegel mit Spitze s ist $0_{ik} = (sy)_{ik}$, also

$$E_0 = (1\,sy\,\pi^2)\,(1u') = -\tfrac{1}{2}\,(syy'\,\pi^2)\,.\,(su').$$

Wir berechnen E_0 für die Gerade φ_{ik} von (60). Nach (62) wird:

$$E_0(\varphi) = 3(\varphi HH_2\pi^2)(\varphi u') + \lambda\,.\,(\varphi 1^2\pi^2)(\varphi u') + \lambda'\,.\,(\varphi 0^2\pi^2)(\varphi u').$$

Bringt man hier die beiden Reihen φ in den Klammerfaktor und setzt

$$\varphi_{ik} = \tfrac{3}{2}(HH_1)_{ik} + \lambda\,.\,0_{ik},$$

so kommt

$$\left.\begin{aligned} E_0(\varphi) = (Hu')\,.\,\{\tfrac{9}{2}(HH_1H_2\pi^2) - \tfrac{9}{4}\lambda\pi_{02,04} - \tfrac{3}{4}\lambda\pi_{02,22} - \lambda'\pi_{02,03}\} + \\ + \lambda(H_1u')\pi_{02,03} + \lambda^2 E_0 \end{aligned}\right\} \quad . \qquad (90)$$

Hier kann man noch $(HH_1H_2\pi^2)$ nach (84) ausdrücken.

Setzen wir in (90) $\lambda = 0$, so geht φ_{ik} über in die Tangente $(HH_1)_{ik}$ an C_H im Heftpunkte H; E_0 zerfällt demgemäss in (Hu') und die dazugehörige Schmiegebene h_{ijk}, Gleichung (84).

Bei $\lambda = 8Q$ wird φ_{ik} mit m_{ik} identisch. Da auch die Geraden m_{ik} eine abwickelbare Regelfläche bilden, muss auch für diesen Fall $E_0(m)$ in zwei Faktoren zerfallen, von denen der eine den durch (42) dargestellten Punkt M, der andere die zugehörige Schmiegebene geben muss. Also muss nach (32) der zweite Faktor die Form $\pi_{02,03}$ sein.

Das dies in der Tat der Fall ist, kann man verifizieren, wenn man von einer Identität Gebrauch macht, die wir jetzt ableiten wollen. Setzt man in (90) $\lambda = 8Q$, so entsteht im letzten Term $64Q^2E_0$ und es lässt sich QE_0 durch andere Komitanten ausdrücken.

Es sei $\dot{0}_{ik}$ mit 0_{ik} äquivalent. Dann können wir in dem Produkt

$$Z = 0_{ik,\alpha\beta}\,\dot{0}_{\gamma\delta,\varepsilon\eta} = (0\,i^2k^2)\,\dot{0}_{\gamma\delta}\,0_{\alpha\beta}\,\dot{0}_{\varepsilon\eta}$$

die Reihe $\dot{0}$ des zweiten Faktors in den Klammerfaktor bringen und erhalten

$$Z = 0_{ik,\varepsilon\eta}\,0_{\gamma\delta,\alpha\beta} + 2(\dot{0}0\,ik^2)\,\dot{0}_{\varepsilon\eta}\,i_{\gamma\delta}\,0_{\alpha\beta} + 2(\dot{0}0\,i^2k)\,\dot{0}_{\varepsilon\eta}\,k_{\gamma\delta}\,0_{\alpha\beta}.$$

Hier werden die beiden letzten Terme, wenn wir auch die zweite Reihe $\dot{0}$ in den Klammerfaktor schaffen, wegen $(\dot{0}^2 0) = 0$ gleich

$$+\,i_{0k,\gamma\delta}\,0_{\varepsilon\eta,\alpha\beta} + k_{0i,\gamma\delta}\,0_{\varepsilon\eta,\alpha\beta},$$

und dies gibt wegen der zyklischen Symmetrie von $i_{0k,\gamma\delta}$ bezgl. $i, 0, k$:

$$-\,0_{ik,\gamma\delta}\,.\,0_{\varepsilon\eta,\alpha\beta}.$$

Also wird

$$Z = 0_{ik,\varepsilon\eta}\,0_{\gamma\delta,\alpha\beta} + 0_{ik,\gamma\delta}\,0_{\varepsilon\eta,\alpha\beta}$$

und damit erhält man die Identität:

$$0_{ik,\alpha\beta}\,0_{\gamma\delta,\varepsilon\eta} + 0_{ik,\gamma\delta}\,0_{\varepsilon\eta,\alpha\beta} + 0_{ik,\varepsilon\eta}\,0_{\alpha\beta,\gamma\delta} = 0, \quad . \; . \; . \quad (91)$$

d.h. wir haben im Produkte Z zyklische Symmetrie in den drei Paaren $\alpha\beta, \gamma\delta$ und $\varepsilon\eta$.

Bei der Ableitung von (91) können statt dieser Paare auch Reihen u', v', w' stehen. Steht z.B. u' statt $\varepsilon\eta$, so haben wir

$$0_{ik,\alpha\beta} \,.\, 0_{\gamma\delta} (0u') = -0_{ik,\gamma\delta} \,.\, (0u')\, 0_{\alpha\beta} - 0_{ik} (0u') \,.\, 0_{\alpha\beta,\gamma\delta}$$

Dies kann man anwenden für den Fall

$$0_{ik,\alpha\beta} \,.\, 0_{\gamma\delta} (0u') = 0_{13,23} \,.\, 0_{1\pi} (0u') = Q \,.\, E_0$$

und erhält

$$Q E_0 = -0_{13,1\pi} \,.\, (0u')\, 0_{23} - 0_{13} (0u') \,.\, 0_{23,1\pi}$$

oder schliesslich

$$Q E_0 = -\tfrac{1}{2}\, \pi_{02,03} \,.\, (0u')\, 0_{23} + \tfrac{3}{16} (Hu') \,.\, \pi_{02,04} \quad . \quad . \quad . \quad . \quad (92)$$

§ 11.

Konstruiert man zu vier Erzeugenden von F die assoziierte Gerade G_5 und lässt dann die vier Erzeugenden auf 0_{ik} zusammenrücken, so geht für $Q \neq 0$ G_5 in die zur Erzeugenden 0_{ik} assoziierte Gerade, oder, wie wir kurz sagen wollen, in die „Bei-Gerade" g_{ik} über. Ich habe vor Kurzem bewiesen ¹), dass g_{ik} gegeben wird durch

$$g_{ik} = 0_{ik} \,.\, P - 1_{ik} \,.\, 12\, Q\, (3\, Q s + R^2) + 2_{ik} \,.\, 36\, Q^2 R - 3_{ik} \,.\, 72\, Q^3 . \quad (93)$$

Hierbei sind Q, R, S und T durch (41) gegeben und P ist der Ausdruck

$$P = 2 R^3 + 12\, Q R S + 9\, Q^2 T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

Er ist vom φ'-Gewichte $\varrho = 30$ und vom λ-Gewicht $\sigma = 15$.

Setzen wir in der Gleichung $\pi_{02,03} = 0$ der Heftebene $\pi_{ik} = g_{ik}$, so entsteht Null, *die Beigerade schneidet also die Heftebene*. Wir nennen den Schnittpunkt G. Seine Gleichung findet man, wenn man einen nicht in Tangentialraum M'_{02} liegenden Punkt mit der Heftebene zu einem R_3 verbindet und diesem dann mit g_{ik} schneidet. Man erhält bei $Q \neq 0$:

$$(Gu') = R \,.\, (Hu') + 4\, Q \,.\, (3u')\, 3_{02} = 0 \quad . \quad . \quad . \quad . \quad . \quad (95)$$

Wir haben dann ausser dem Heftpunkt H in der Heftebene m_{ijk} zwei invariante Punkte: G und den durch (42) gegebenen Punkt M. Hiermit lässt sich auf der Erzeugenden 0_{ik} selbst ein zweiter invarianter Punkt F konstruieren, nämlich der Schnittpunkt von 0_{ik} mit der Verbindungslinie GM. Als Gleichung von F erhält man:

$$F = M + G = (Hu')\, (2 R + \tfrac{3}{5}\, 2_{05,03}) - 16\, Q \,.\, (0u')\, 0_{23} \quad . \quad . \quad (96)$$

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¹) Diese Proceedings **43**, 325—333 (1940).

oder auch, nach (82):

$$(u'F) = (Hu') \,.\, (\tfrac{3}{2} R + 3\,Q') - 16\,Q \,.\, (0u')\, 0_{23} = 0 \quad . \quad . \qquad (96a)$$

F, G und M sind Differentialkontravarianten mit den Gewichten $\varrho = 14$ und $\sigma = 8$.

Die Gerade

$$\varphi_{ik} = \tfrac{3}{2} (HH_1)_{ik} + \lambda \,.\, 0_{ik}$$

im Büschel um H in der Heftebene lässt sich dann durch die beiden Basisgeraden $(HM)_{ik}$ und $(HG)_{ik}$ wie folgt darstellen

$$\varphi_{ik} = \frac{1}{64\,Q^2} \{(HM)_{ik} \,.\, (3\lambda - 8\,Q) + (HG)_{ik} \,.\, (3\lambda - 24\,Q)\} \quad . \qquad (97)$$

Hieraus folgt z.B. dass die Gerade MG die Tangente $h_{ik} = (HH_1)_{ik}$ im Punkte $M + 3G$, die Heftgerade a_{ik} aber im Punkte $M - 3G$ schneidet.

Die Beigerade g_{ik} gibt mit der Erzeugenden 0_{ik} verbunden den kovarianten R_3

$$(B'x) = R \,.\, x_{02} - 2\,Q \,.\, x_{03} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (98)$$

und zwei aufeinander folgende dieser Räume schneiden sich in der Ebene

$$\left.\begin{aligned} (\tfrac{2}{3} R^2 + 2\,QR' - 2\,Q'R) \,.\, \pi_{02,03} - 2\,QR \,.\, (\pi_{02,13} + \pi_{02,04}) + \\ + 4\,Q^2 \,.\, (\pi_{03,13} + \pi_{03,04}) = 0. \end{aligned}\right\} . \qquad (99)$$

Man rechnet leicht nach, dass diese Ebene auch die Beigerade schneidet.

Mathematics. — *Beiträge zur Theorie der Systeme* PFAFF*scher Gleichungen. IV. Beweis des Haupttheorems für den Fall, dass der Rang einen beliebigen Wert hat.* Von J. A. SCHOUTEN und W. VAN DER KULK.

(Communicated at the meeting of April 27, 1940.)

1. *Konstruktion der Anfangswerte* $\overset{1}{p}_0, \dots, \overset{\varrho+1}{p}_0$.

Der Teil (a) des Haupttheorems ist bewiesen für $p = 2\varrho$ und $p = 2\varrho + 1$ [1]. Nehmen wir also an es sei $p \geqq 2\varrho + 2$ und ausserdem, dass der Beweis schon geliefert sei für einen um eins geringeren Wert von p und denselben Wert von ϱ. Durch umnumerieren kann man stets erreichen dass (III 7) übergeht in

$$Q_{[23} \dots Q_{2\varrho\, 2\varrho+1]} \neq 0; \;\; Q_{ba} \overset{\text{def}}{=} 2\, B^{\omega}_{[b}\, (\partial_{|\omega|}\, B^{\lambda}_{a]})\, Q_\lambda. \quad . \quad . \quad . \qquad (1)$$

(III 8) ist gleichbedeutend mit

$$P_{a_1 \dots a_\varrho} \overset{\text{def}}{=} B^{\lambda_1}_{a_1} \dots B^{\lambda_\varrho}_{a_\varrho}\, P_{\lambda_1 \dots \lambda_\varrho} \neq 0; \;\; a_1, \dots, a_\varrho = 1, \dots, \mathrm{p}, \; . \; . \; . \qquad (2)$$

und (III 9) mit

$$Q_{[ab}\, P_{a_1 \dots a_\varrho]} = 0; \;\; a, b, a_1, \dots, a_\varrho = 1, \dots, \mathrm{p}. \quad . \quad . \quad . \qquad (3)$$

Wir beweisen jetzt, dass nicht *alle* Bestimmungszahlen $P_{b_1 \dots b_\varrho}$; $b_1, \dots, b_\varrho = 2, \dots, 2\varrho + 1$ verschwinden können. Dazu betrachten wir in dem Gebiet der Indizes $2, \dots, 2\varrho + 1$ die einfachen $(\varrho - 1)$-Vektoren

$$P_{b_2 \dots b_\varrho (k)} \quad ; \quad k = 1, 2\varrho + 2, \dots, \mathrm{p}.\ ^{2)} \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Wäre nun die Behauptung unrichtig, so würde aus (3) für $a_1 = k$, $a_2 = b_2, \dots, a_\varrho = b_\varrho$; $a, b, b_2, \dots, b_\varrho = 2, \dots, 2\varrho + 1$ folgen

$$Q_{[ab}\, P_{b_2 \dots b_\varrho] (k)} = 0 \quad ; \quad k = 1, 2\varrho + 2, \dots, \mathrm{p}. \quad . \quad . \quad . \qquad (5)$$

Nun ist aber Q_{ab} laut (1) im Gebiete der Indizes $2, \dots, 2\varrho + 1$ vom höchsten Range. Infolge des Hilfssatzes (III, Fussnote S. 455) wäre also

$$P_{b_2 \dots b_\varrho k} = 0; \;\; b_2, \dots, b_\varrho = 2, \dots, 2\varrho + 1; \;\; k = 1, 2\varrho + 2, \dots, \mathrm{p}, \qquad (6)$$

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[1] Beiträge zur Theorie der Systeme PFAFFscher Gleichungen, III, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 453—462 (1940), hier weiter zitiert als III.

[2] Die Abdrosselung (k) deutet an, dass wir nicht einen ϱ-Vektor, sondern $p - 2\varrho$ $(\varrho - 1)$-Vektoren betrachten.

und infolge der gemachten Annahme auch

$$P_{a_1 b_2 \ldots b_\varrho} = 0 \quad ; \quad b_2, \ldots, b_\varrho = 2, \ldots, 2\varrho + 1 \quad ; \quad a_1 = 1, \ldots, p. \qquad (7)$$

Jetzt betrachten wir die einfachen $(\varrho - 2)$-Vektoren

$$P_{b_3 \ldots b_\varrho (k)(l)} \quad ; \quad b_3, \ldots, b_\varrho = 2, \ldots, 2\varrho + 1 \quad ; \quad k, l = 1, 2\varrho + 2, \ldots, p. \qquad (8)$$

Aus (3) und (7) würde dann für $a = k$, $a_1 = l$; $a_2 = b_2, \ldots, a_\varrho = b_\varrho$ folgen

$$\left. \begin{array}{r} Q_{[b b_2} P_{b_3 \ldots b_\varrho](k)(l)} = 0 \;;\; k, l = 1, 2\varrho + 2, \ldots, p \;; \\ b, b_2, \ldots, b_\varrho = 2, \ldots, 2\varrho + 1. \end{array} \right\} \quad . \quad . \quad (9)$$

und daraus ginge unter Verwendung desselben Hilfssatzes hervor

$$P_{a_1 a_2 b_3 \ldots b_\varrho} = 0 \;;\; b_3, \ldots, b_\varrho = 2, \ldots, 2\varrho + 1 \;;\; a_1, a_2 = 1, \ldots, p. \qquad (10)$$

In dieser Weise fortfahrend liesse sich schliesslich dartun dass $P_{a_1 \ldots a_\varrho} = 0$ wäre; $a_1, \ldots, a_\varrho = 1, \ldots, p$ und dies ist im Gegensatz zu (2). Nicht alle Bestimmungszahlen $P_{a_1 \ldots a_\varrho}$ mit Indizes $2, \ldots, 2\varrho + 1$ verschwinden also und durch eine Umnummerierung, die auf (1) keinen Einfluss hat, lässt sich demnach stets erreichen, dass

$$P_{2 \ldots \varrho+1} \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

ist.

In der X_{n-1} $x^1 = \underset{0}{x^1}$, mit den Koordinaten x^α; $\alpha, \beta = 2, \ldots, n$, betrachten wir jetzt die Grössen

$$\left. \begin{array}{l} \overset{0}{B}{}^\alpha_b (x^2, \ldots, x^n) \overset{\text{def}}{=} B^\alpha_b (\underset{0}{x^1}, x^2, \ldots, x^n) \;;\; b = 2, \ldots, p \;;\; \alpha, \beta = 2, \ldots, n \\ \overset{0}{C}{}^\varkappa_\beta (x^2, \ldots, x^n) \overset{\text{def}}{=} C^\varkappa_\beta (\underset{0}{x^1}, x^2, \ldots, x^n) \;;\; \varkappa = p + 1, \ldots, n. \end{array} \right\} (12)$$

Da infolge (III 10) [3])

$$\overset{0}{B}{}^\alpha_b \, \overset{0}{C}{}^\varkappa_\alpha = 0 \;;\; b = 2, \ldots, p \;;\; \varkappa = p + 1, \ldots, n \quad . \quad . \quad . \qquad (13)$$

ist, sind

$$\overset{0}{C}{}^\varkappa_\beta \, dx^\beta = 0 \;;\; \varkappa = p + 1, \ldots, n. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

und

$$\overset{0}{B}{}^\alpha_b \, \partial_\alpha f = 0 \;;\; b = 2, \ldots, p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

in dieser X_{n-1} adjungierte Systeme. Wir behaupten nun, dass die Komponenten Q_β; $\beta = 2, \ldots, n$ und $P_{\beta_1 \ldots \beta_\varrho}$; $\beta_1, \ldots, \beta_\varrho = 2, \ldots, n$, die in dieser

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[3]) Wir denken uns die $B^\varkappa_b$ wieder in der Form (III 10) geschrieben.

X_{n-1} einen Vektor bzw. ϱ-Vektor in $x^\alpha = \underset{0}{x^\alpha}$ darstellen, i.b. auf das PFAFFsche System (14) Bedingungen genügen, die für dieses System die Bedeutung von (III 6, 7, 8, 9) haben. Aus (III 6) folgt in $x^\alpha = \underset{0}{x^\alpha}$

$$\overset{0}{B}{}^\beta_b \, Q_\beta = B^\lambda_b \, Q_\lambda - B^1_b \, Q_1 = 0 \; ; \; b = 2, \ldots, \mathrm{p}, \quad . \quad . \quad . \quad (16)$$

da $B^1_b = 0$ ist für $b \neq 1$ infolge (III 10), und dies ist die Gleichung (III 6) für das System (14) in X_{n-1}. Aus demselben Grunde ist

$$\left.\begin{array}{r} 2 \overset{0}{B}{}^\beta_{[b} \, \partial_{|\beta|} \, \overset{0}{B}{}^\alpha_{a]} = 2 \, (B^\beta_{[b} \, \partial_{|\beta|} \, B^\alpha_{a]})_{x^1 = \underset{0}{x^1}} = 2 \, (B^\omega_{[b} \, \partial_{|\omega|} \, B^\alpha_{a]})_{x^1 = \underset{0}{x^1}} \; ; \\ b, \mathrm{a} = 2, \ldots, \mathrm{p} \end{array}\right\} . \quad (17)$$

sodass aus (1) folgt in $x^\alpha = \underset{0}{x^\alpha}$

$$Q_{[23} \ldots Q_{2\varrho \, 2\varrho+1]} \neq 0 \; ; \; Q_{ba} \overset{\mathrm{def}}{=} 2 \overset{0}{B}{}^\beta_{[b} \, (\partial_{|\beta|} \, \overset{0}{B}{}^\alpha_{a]}) \, Q_\alpha \; ; \; \mathrm{a}, b = 2, \ldots, \mathrm{p} \quad (18)$$

und dies ist die Gleichung (III 7) für das System (14) in X_{n-1}. Aus (3) folgt in $x^\alpha = \underset{0}{x^\alpha}$

$$Q_{[ab} \, P_{b_1 \ldots b_\varrho]} = 0 \, ; \, P_{b_1 \ldots b_\varrho} \overset{\mathrm{def}}{=} \overset{0}{B}{}^{\beta_1}_{b_1} \ldots \overset{0}{B}{}^{\beta_\varrho}_{b_\varrho} \, P_{\beta_1 \ldots \beta_\varrho} \, ; \, \mathrm{a}, b, b_1, \ldots, b_\varrho = 2, \ldots, \mathrm{p}, \quad (19)$$

und dies ist die Gleichung (III 9) für das System (14) in X_{n-1}. Aus (12) folgt schliesslich

$$P_{2 \ldots \varrho+1} \neq 0 \; ; \; P_{b_1 \ldots b_\varrho} \overset{\mathrm{def}}{=} \overset{0}{B}{}^{\beta_1}_{b_1}, \ldots \overset{0}{B}{}^{\beta_\varrho}_{b_\varrho} \, P_{\beta_1 \ldots \beta_\varrho} \; ; \; b_1, \ldots, b_\varrho = 2, \ldots, \mathrm{p}, \quad (20)$$

und dies ist die Gleichung (III 8) für das System (14) in X_{n-1}. Ausserdem geht aus (III 3) und (1) hervor, dass das System (14) gerade den Halbrang ϱ hat. Wir haben also jetzt ein PFAFFsches System erhalten mit einem um eins geringeren Wert von p und denselben Wert von ϱ. Da wir angenommen haben, dass der Beweis für ein solches System schon geliefert sei, existiert also ein System von $\varrho + 1$ Funktionen $\overset{1}{p}_0, \ldots, \overset{\varrho+1}{p}_0$, die Lösungen sind der Gleichungen

$$\left\| \begin{array}{ccc} \overset{0}{X}_2 \overset{1}{p}_0 & \ldots & \overset{0}{X}_2 \overset{\varrho+1}{p}_0 \\ \vdots & & \vdots \\ \overset{0}{X}_\mathrm{p} \overset{1}{p}_0 & \ldots & \overset{0}{X}_\mathrm{p} \overset{\varrho+1}{p}_0 \end{array} \right\| = \text{Matrix vom Range } \varrho \, ; \quad \overset{0}{X}_b = \overset{0}{B}{}^\beta_b \, \partial_\beta ; \; b = 2, \ldots, \mathrm{p} ; \; \beta = 2, \ldots, n, \quad (21)$$

und ausserdem in $x^\alpha = \underset{0}{x^\alpha}$ noch den Gleichungen

$$(\partial_{[\beta_1} \overset{1}{p}_0) \ldots (\partial_{\beta_{\varrho+1}]} \overset{\varrho+1}{p}_0) = Q_{[\beta_1} P_{\beta_2 \ldots \beta_{\varrho+1}]} \, ; \, \beta_1, \ldots, \beta_{\varrho+1} = 2, \ldots, n \quad . \quad (22)$$

genügen.

2. *Konstruktion der Funktionen* $\overset{1}{p}, \ldots, \overset{\varrho+1}{p}$.

Das System (III 4) ist gleichwertig mit

$$\left.\begin{aligned} &\text{a)} \quad X_1 \overset{i}{p} = \overset{2}{\sigma} X_2 \overset{i}{p} + \ldots + \overset{\varrho+1}{\sigma} X_{\varrho+1} \overset{i}{p}; \ i = 1, \ldots, \varrho+1 \\ &\text{b)} \quad \theta_k \overset{\text{def}}{=} (X_k \overset{[1}{p})(X_2 \overset{2}{p}) \ldots (X_{\varrho+1} \overset{\varrho+1]}{p}) = 0; \ k = \varrho+2, \ldots, \mathrm{p}, \end{aligned}\right\} \quad . \ (23)$$

vorausgesetzt, dass die Ungleichung

$$(X_2 \overset{[i_2}{p}) \ldots (X_{\varrho+1} \overset{i_{\varrho+1}]}{p}) \neq 0; \ i_2, \ldots, i_{\varrho+1} = 1, \ldots, \varrho+1 \ . \quad . \quad . \quad (24)$$

erfüllt sei.

Sind $\overset{1}{p}, \ldots, \overset{\varrho+1}{p}$ Lösungen von (23*a*), so gilt für die mit diesen Lösungen nach (23*b*, def.) konstruierten Funktionen θ_k

$$X_1 \theta_k = X_2 (\overset{2}{\sigma} \theta_k) + \ldots + X_{\varrho+1} (\overset{\varrho+1}{\sigma} \theta_k) + \zeta_k; \ k = \varrho+2, \ldots, \mathrm{p} \quad . \quad (25)$$

wo

$$\zeta_k = (\varrho+1) \{ p_{1[k,2\ldots\varrho+1]} - \overset{2}{\sigma} p_{2[k,2\ldots\varrho+1]} - \ldots - \overset{\varrho+1}{\sigma} p_{\varrho+1[k,2\ldots\varrho+1]} \} \quad . \quad (26)$$

und

$$p_{ab,a_1\ldots a_\varrho} \overset{\text{def}}{=} 2 B^{\omega}_{[a} (\partial_{\omega} B^{\lambda}_{b}) B^{\lambda_1}_{a_1} \ldots B^{\lambda_\varrho}_{a_\varrho]} (\partial_{[\lambda} \overset{1}{p}) (\partial_{\lambda_1} \overset{2}{p}) \ldots (\partial_{\lambda_\varrho]} \overset{\varrho+1}{p}) \ . \quad (27)$$

ist. Da die ζ_k für solche Lösungen von (23*a*), die auch (23*b*) genügen, verschwinden müssen, benutzen wir die Gleichungen

$$\zeta_l = 0; \ l = \varrho+2, \ldots, 2\varrho+1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

dazu, $\overset{2}{\sigma}, \ldots, \overset{\varrho+1}{\sigma}$ zu lösen. (Wir benutzen also hier nur die ersten ϱ der ζ_k). Für jede dieser Grössen finden wir aus (28) eine rationale Funktion der ersten Ableitungen von $\overset{1}{p}, \ldots, \overset{\varrho+1}{p}$ nach $x^2, \ldots, x^n$ (also *nicht* nach x^1), deren Koeffizienten in der Umgebung von $x^\varkappa = \underset{0}{x^\varkappa}$ reguläre Funktionen von $x^1, \ldots, x^n$ sind. Der Nenner ist für jedes σ

$$\triangle = \begin{vmatrix} p_{2[\varrho+2,2\ldots\varrho+1]} & \ldots & p_{\varrho+1[\varrho+2,2\ldots\varrho+1]} \\ \vdots & & \vdots \\ p_{2[2\varrho+1,2\ldots\varrho+1]} & \ldots & p_{\varrho+1[2\varrho+1,2\ldots\varrho+1]} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Substituiert man diese Werte von $\overset{2}{\sigma}, \ldots, \overset{\varrho+1}{\sigma}$ in (23*a*), dann entstehen nach Umrechnung Gleichungen von der Form

$$\partial_1 \overset{i}{p} = \overset{i}{F} (x^1, \ldots, x^n, \partial_\beta \overset{1}{p}, \ldots, \partial_\beta \overset{\varrho+1}{p}); \ \beta = 2, \ldots, n; \ i = 1, \ldots, \varrho+1 \ . \ (30)$$

Die $\overset{i}{F}$ sind rationale Funktionen der Variabeln $\partial_\beta \overset{1}{p}, \ldots, \partial_\beta \overset{\varrho+1}{p}$; $\beta = 2, \ldots, n$ mit Koeffizienten, die in der Umgebung von $x^\varkappa = \underset{0}{x}^\varkappa$ reguläre Funktionen von $x^1, \ldots, x^n$ sind. Der Nenner ist für alle $\overset{i}{F}$ gleich $\triangle$. Wir betrachten diese Funktionen für die Argumentwerte

$$x^\varkappa = \underset{0}{x}^\varkappa; \; \partial_\beta \overset{i}{p} = (\partial_\beta \overset{i}{p}_0)_{x^\alpha = \underset{0}{x}^\alpha}; \; \beta = 2, \ldots, n; \; i = 1, \ldots, \varrho + 1. \quad . \quad (31)$$

Infolge (22) und (29) ist für diese Argumentwerte

$$\triangle = (\varrho + 1)^{-\varrho} \begin{vmatrix} Q_{2[\varrho+2} \; P_{2\ldots\varrho+1]} & \ldots & Q_{\varrho+1\,[\varrho+2} \; P_{2\ldots\varrho+1]} \\ \vdots & & \vdots \\ Q_{2[2\varrho+1} \; P_{2\ldots\varrho+1]} & \ldots & Q_{\varrho+1\,[2\varrho+1} \; P_{2\ldots\varrho+1]} \end{vmatrix}, \quad . \quad . \quad . \quad (32)$$

und aus dieser Gleichung folgt mit Hilfe der (III S.) angewandten Umrechnung

$$\triangle^2 = (2\varrho + 2)^{-2\varrho} \left(\frac{(2\varrho)!}{\varrho!} \right)^2 (P_{2\ldots\varrho+1})^{2\varrho} (Q_{[23} \ldots Q_{2\varrho\, 2\varrho+1]})^2 \quad . \quad . \quad (33)$$

woraus unter Berücksichtigung von (18) und (20) hervorgeht, dass in der Umgebung dieser Argumentwerte $\triangle \neq 0$ ist. In dieser Umgebung sind also die $\overset{i}{F}$ regulär analytische Funktionen von $x^1, \ldots, x^n$; $\partial_\beta \overset{i}{p}$; $\beta = 2, \ldots, n$; $i = 1, \ldots, \varrho + 1$.

Nach Cauchy—Kowalewski *existieren also* $\varrho + 1$ *Funktionen* $\overset{1}{p}, \ldots, \overset{\varrho+1}{p}$ *der Variabeln* $x^1, \ldots, x^n$, *die* (23a) *erfüllen und für die ausserdem für* $x^1 = \underset{0}{x}^1$ *gilt*

$$\overset{i}{p}(\underset{0}{x}^1, x^2, \ldots, x^n) = \overset{i}{p}_0 (x^2, \ldots, x^n); \; i = 1, \ldots, \varrho + 1 \quad . \quad . \quad . \quad (34)$$

In $x^\varkappa = \underset{0}{x}^\varkappa$ ist infolge (22) und (34)

$$(\partial_{[\beta_1} \overset{1}{p}) \ldots (\partial_{\beta_{\varrho+1}]} \overset{\varrho+1}{p}) = Q_{[\beta_1} P_{\beta_2 \ldots \beta_{\varrho+1}]}; \; \beta_1, \ldots, \beta_{\varrho+1} = 2, \ldots, n \,. \quad (35)$$

Ueberschiebung mit $B_2^{\beta_2} \ldots B_{\varrho+1}^{\beta_{\varrho+1}}$ ergibt wegen (III 6) in $x^\varkappa = \underset{0}{x}^\varkappa$

$$(\partial_{\beta_1} \overset{[1}{p}) (B_2^{|\beta_2|} \partial_{\beta_2} \overset{2}{p}) \ldots (B_{\varrho+1}^{|\beta_{\varrho+1}|} \partial_{\beta_{\varrho+1}} \overset{\varrho+1]}{p}) = \frac{1}{\varrho + 1} Q_{\beta_1} P_{2\ldots\varrho+1}, \quad . \quad (36)$$

und, da $B_2^1 = 0, \ldots, B_{\varrho+1}^1 = 0$ ist, folgt in $x^\varkappa = \underset{0}{x}^\varkappa$

$$(\partial_{\beta_1} \overset{[1}{p}) (X_2 \overset{2}{p}) \ldots (X_{\varrho+1} \overset{\varrho+1]}{p}) = \frac{1}{\varrho + 1} Q_{\beta_1} P_{2\ldots\varrho+1} \quad . \quad . \quad . \quad (37)$$

Infolge (1) und (11) ist nun das rechte Glied, und somit auch das linke, für $x^\varkappa = \underset{0}{x}^\varkappa$ nicht Null. Das linke Glied ist also auch in einer Umgebung von $x^\varkappa = \underset{0}{x}^\varkappa$ nicht Null und in dieser Umgebung gilt daher

$$(X_2 \overset{[i_2}{p}) \dots (X_{\varrho+1} \overset{i_{\varrho+1}]}{p}) \neq 0;\ i_2, \dots, i_{\varrho+1} = 1, \dots, \varrho+1 \quad . \quad . \quad (38)$$

womit bewiesen ist, dass die Lösungen $\overset{1}{p}, \dots, \overset{\varrho+1}{p}$ auch die Gleichungen (24) erfüllen.

3. *Beweis, dass* $\overset{1}{p}, \dots, \overset{\varrho+1}{p}$ *auch* (23*b*), (III 4) *und* (III 5) *erfüllen.*

Wir benutzen die abgekürzte Bezeichnung (vergl. II 8*b*)

$$\Psi_{a,b} \overset{\text{def}}{=} (\varrho+1)\, p_{a\,[b,2\dots\varrho+1]}\,; \quad a, b = 1, \dots, \mathrm{p} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

welcher Ausdruck mit den im vorigen Paragraphen bestimmten $\overset{1}{p}, \dots, \overset{\varrho+1}{p}$ zu bilden ist. Sodann verwenden wir die Gleichungen (26) in der Form

$$\left.\begin{array}{lllll}
\zeta_{\varrho+2} & = \Psi_{1,\varrho+2} & - \overset{2}{\sigma}\, \Psi_{2,\varrho+2} & - \dots - & \overset{\varrho+1}{\sigma}\, \Psi_{\varrho+1,\varrho+2} \\
\vdots & \quad\vdots & & & \\
\zeta_{2\varrho+1} & = \Psi_{1,2\varrho+1} & - \overset{2}{\sigma}\, \Psi_{2,2\varrho+1} & - \dots - & \overset{\varrho+1}{\sigma}\, \Psi_{\varrho+1,2\varrho+1} \\
\zeta_j & = \Psi_{1,j} & - \overset{2}{\sigma}\, \Psi_{2,j} & - \dots - & \overset{\varrho+1}{\sigma}\, \Psi_{\varrho+1,j}
\end{array}\right\} \quad . \quad . \quad (40)$$

wo j einen Index von $2\varrho+2$ bis p darstellt, um $\overset{2}{\sigma}, \dots, \overset{\varrho+1}{\sigma}$ zu eliminieren. Da $\zeta_{\varrho+2}, \dots, \zeta_{2\varrho+1}$ verschwinden, ergibt sich

$$\Delta\, \zeta_j = \begin{vmatrix} \Psi_{1,\varrho+2} & \dots & \Psi_{\varrho+1,\varrho+2} \\ \vdots & & \vdots \\ \Psi_{1,2\varrho+1} & \dots & \Psi_{\varrho+1,2\varrho+1} \\ \Psi_{1,j} & \dots & \Psi_{\varrho+1,j} \end{vmatrix} \overset{\text{def}}{=} D_j \quad . \quad . \quad . \quad . \quad (41)$$

für $j = 2\varrho+2, \dots, \mathrm{p}$. In der Umgebung von $x^\varkappa = \underset{0}{x}^\varkappa$ ist $\Delta \neq 0$. Es gilt die D_j umzuformen.

Dazu bilden wir aus den Lösungen $\overset{1}{p}, \dots, \overset{\varrho+1}{p}$ einen Vektor

$$w_\lambda \overset{\text{def}}{=} \underset{1}{\mu}\, \partial_\lambda \overset{1}{p} + \dots + \underset{\varrho+1}{\mu}\, \partial_\lambda \overset{\varrho+1}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

und wählen $\underset{1}{\mu}, \dots, \underset{\varrho+1}{\mu}$ so, dass

$$B_b^\lambda\, w_\lambda = 0; \quad b = 2, \dots, \varrho+1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

ist und nicht alle μ verschwinden. Dann verschwindet auf Grund von (22) und (34) auch w_λ nicht.

Wegen (23*a*), (42) und (43) ist auch

$$B_1^{\lambda} w_\lambda = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{44}$$

Sodann existiert ein einfacher ϱ-Vektor $\pi_{\lambda_1 \ldots \lambda_\varrho}$, so dass

$$(\partial_{[\lambda_1} \overset{1}{p}) \ldots . (\partial_{\lambda_{\varrho+1}]} \overset{\varrho+1}{p}) = w_{[\lambda_1} \pi_{\lambda_2 \ldots \lambda_{\varrho+1}]} \quad . \quad . \quad . \quad . \quad . \tag{45}$$

ist, da w_λ Teiler des Gradientproduktes zur Linken ist. Wird diese Gleichung überschoben mit $B_2^{\lambda_2} \ldots B_{\varrho+1}^{\lambda_{\varrho+1}}$, so folgt wegen (43)

$$(\varrho + 1) B_2^{\lambda_2} \ldots . B_{\varrho+1}^{\lambda_{\varrho+1}} (\partial_{[\lambda_1} \overset{1}{p}) \ldots . (\partial_{\lambda_{\varrho+1}]} \overset{\varrho+1}{p}) = w_{\lambda_1} \pi_{2 \ldots \varrho+1} \quad . \quad . \tag{46}$$

wenn wir abkürzend schreiben

$$\pi_{a_1 \ldots a_\varrho} \overset{\text{def}}{=} B_{a_1}^{\lambda_1} \ldots . B_{a_\varrho}^{\lambda_\varrho} \pi_{\lambda_1 \ldots \lambda_\varrho}; \; a_1, \ldots, a_\varrho = 1, \ldots, p. \quad . \quad . \tag{47}$$

Nun gilt für die linke Seite von (46)

$$\left.\begin{aligned} B_2^{\lambda_2} \ldots B_{\varrho+1}^{\lambda_{\varrho+1}} (\partial_{[\lambda_1} \overset{1}{p}) \ldots (\partial_{\lambda_{\varrho+1}]} \overset{\varrho+1}{p}) = \\ = B_2^{\lambda_2} \ldots B_{\varrho+1}^{\lambda_{\varrho+1}} (\partial_{\lambda_1} \overset{[1}{p}) \ldots (\partial_{\lambda_{\varrho+1}} \overset{\varrho+1]}{p}) = (\partial_{\lambda_1} \overset{[1}{p}) (X_2 \overset{2}{p}) \ldots X_{\varrho+1} \overset{\varrho+1]}{p}) \end{aligned}\right\} \tag{48}$$

und dieser Ausdruck ist infolge (37) u.f. ungleich Null. Also ist

$$\pi_{2 \ldots \varrho+1} \neq 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{49}$$

Aus (23*b*, def.), (43) und (45) folgt

$$\theta_k = \frac{1}{\varrho + 1} w_k \pi_{2 \ldots \varrho+1} \; ; \; w_k \overset{\text{def}}{=} B_k^{\lambda} w_\lambda \; ; \; k = \varrho + 2, \ldots, p. \quad . \tag{50}$$

Aus (39) und (45) folgt

$$\left.\begin{aligned} \Psi_{a,b} = (\varrho + 1) B_{a[b}^{\cdot\cdot\lambda_1} B_2^{\lambda_2} \ldots B_{\varrho+1]}^{\lambda_{\varrho+1}} w_{[\lambda_1} \pi_{\lambda_2 \ldots \lambda_{\varrho+1}]} = \\ = w_{a[b} \pi_{2 \ldots \varrho+1]} (\mathit{mod.}\ w_b) \; ; \; w_{ab} \overset{\text{def}}{=} B_{ab}^{\cdot\cdot\lambda} w_\lambda \\ B_{ab}^{\cdot\cdot\varkappa} \overset{\text{def}}{=} 2 B_{[a}^{\omega} \partial_{|\omega|} B_{b]}^{\varkappa} \\ a = 1, \ldots, p \; ; \; b = \varrho + 2, \ldots, p \end{aligned}\right\} \quad . \quad . \tag{51}$$

und also infolge (49) und (50)

$$\Psi_{a,b} = w_{a[b} \pi_{2 \ldots \varrho+1]} (\mathit{mod.}\ \theta_b) \; ; \; a = 1, \ldots, p \; ; \; b = \varrho + 2, \ldots, p. \tag{52}$$

Setzen wir diese Ausdrücke in (41) ein, so ergibt sich

$$D_j = \begin{vmatrix} w_{1[\varrho+2}\,\pi_{2\ldots\varrho+1]} & \cdots & w_{\varrho+1[\varrho+2}\,\pi_{2\ldots\varrho+1]} \\ \vdots & & \\ w_{1[2\varrho+1}\,\pi_{2\ldots\varrho+1]} & \cdots & w_{\varrho+1[2\varrho+1}\,\pi_{2\ldots\varrho+1]} \\ w_{1[j}\,\pi_{2\ldots\varrho+1]} & \cdots & w_{\varrho+1[j}\,\pi_{2\ldots\varrho+1]} \end{vmatrix} (mod.\, \theta_{\varrho+2},\ldots,\theta_{2\varrho+1}, \theta_j) \quad (53)$$

Wir betrachten jetzt die Determinante rechts für sich und schreiben diese W_j. W_j ist ein homogenes Polynom in den Variablen $w_{ab} = -w_{ba}$; $a,\ b = 1, \ldots,\ 2\varrho + 1, j$. In derselben Weise wie in (III) bei der Ableitung von (III 34) beweist man, dass

$$\left.\begin{aligned} W_j^2 = A^2\{w_{[12}\ldots w_{2\varrho-1\,2\varrho}\,w_{2\varrho+1\,j]}\}^2 \;;\; w_{ab} \overset{\text{def}}{=} B_{\dot a \dot b}{}^{\lambda}\, w_\lambda \\ a, b = 1, \ldots, \mathrm{p} \end{aligned}\right\} \quad (54)$$

wo

$$A = (2\varrho + 2)^{-\varrho-1}\frac{(2\varrho+2)!}{(\varrho+1)!}(\pi_{2\ldots\varrho+1})^{\varrho+1} \quad . \quad . \quad . \quad . \quad (55)$$

ist, für alle Werte von w_{ab}; $a, b = 2, \ldots,\ 2\varrho + 1, j$, die den linearen Gleichungen

$$L_{ab} \overset{\text{def}}{=} w_{[ab}\,\pi_{2\ldots\varrho+1]} = 0 \;;\; a, b = \varrho + 2, \ldots, 2\varrho + 1, j \quad . \quad . \quad (56)$$

genügen. Demzufolge ist

$$\left.\begin{aligned} W_j^2 - A^2\{w_{[12}\ldots w_{2\varrho-1\,2\varrho}\,w_{2\varrho+1\,j]}\}^2 = 0\,(mod.\, L_{ab})\,; \\ a, b = \varrho + 2, \ldots, 2\varrho + 1, j \end{aligned}\right\} \quad . \quad . \quad (57)$$

für *beliebige* Werte von w_{ab} ; $a, b = 2, \ldots.\ 2\varrho + 1, j$, und daraus geht hervor, dass es eine Gleichung gibt von der Form

$$W_j = B\, w_{[12}\ldots w_{2\varrho-1\,2\varrho}\,w_{2\varrho+1\,j]}\,(mod.\, L_{ab})\,;\, a, b = \varrho + 2, \ldots, 2\varrho + 1, j \qquad (58)$$

Insbesondere gilt also

$$\left.\begin{aligned} D_j = B\, w_{[12}\ldots w_{2\varrho-1\,2\varrho}\,w_{2\varrho+1\,j]}\,(mod.\, \theta_{\varrho+2}, \ldots, \theta_{2\varrho+1}, \theta_j, L_{ab}) = \\ = B\, B_{[\dot 1 \dot 2}{}^{\lambda_1}\ldots B_{\dot{2\varrho-1}\,\dot{2\varrho}}{}^{\lambda_\varrho}\,B_{\dot{2\varrho+1}\,\dot j]}{}^{\lambda_{\varrho+1}}\, w_{\lambda_1}\ldots w_{\lambda_{\varrho+1}}(mod.\, \theta_{\varrho+2}, \ldots \\ \ldots, \theta_{2\varrho+1}, \theta_j, L_{ab});\quad (a, b = \varrho + 2, \ldots, 2\varrho + 1. j \end{aligned}\right\} (59)$$

Wegen der besonderen Wahl (III 10) der $B_b^{\varkappa}$ verschwinden nun die ersten p Bestimmungszahlen $B_{\dot a \dot b}{}^{1}, \ldots, B_{\dot a \dot b}{}^{p}$ und der Tensor

$$B_{[\dot 1 \dot 2}{}^{\lambda_1}\ldots B_{\dot{2\varrho-1}\,\dot{2\varrho}}{}^{\lambda_\varrho}\,B_{\dot{2\varrho+1}\,\dot j]}{}^{\lambda_{\varrho+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

liegt also ganz in dem Gebiet der Massvektoren $\underset{p+1}{e}^{\varkappa}, \ldots, \underset{n}{e}^{\varkappa}$. Wegen (III 10) haben die $B_a^{\varkappa}$ die Form

$$B_a^{\varkappa} \overset{*}{=} \underset{a}{e}^{\varkappa} + B_a^{x} \underset{x}{e}^{\varkappa} \; ; \; \mathrm{a} = 1, \ldots, \mathrm{p} \; ; \; x = \mathrm{p}+1, \ldots, \mathrm{n} \;{}^{4}) \; . \quad . \quad (61)$$

und durch lineare Transformation der C_λ^x lässt sich also erreichen, dass

$$C_\lambda^x \overset{*}{=} \overset{x}{e}_\lambda - B_a^x \overset{a}{e}_\lambda \; ; \; x = \mathrm{p}+1, \ldots, \mathrm{n} \; ; \; \mathrm{a} = 1, \ldots, \mathrm{p} \;{}^{4}) \; . \quad . \quad (62)$$

Infolge (III 3) verschwindet der mit $C_{\lambda_1}^{x_1} \ldots C_{\lambda_{\varrho+1}}^{x_{\varrho+1}}$ überschobene Tensor (60). Da dieser Tensor aber in dem Gebiet der Massvektoren $\underset{p+1}{e}^{\varkappa}, \ldots, \underset{n}{e}^{\varkappa}$ liegt, folgt aus (62) dass derselbe Null sein muss, sodass

$$D_j = 0 \, (mod. \; \theta_{\varrho+2}, \ldots, \theta_{2\varrho+1}, \theta_j, L_{ab}) \quad a, b = \varrho+2, \ldots, 2\varrho+1, j \quad (63)$$

Die L_{ab} sind jetzt noch umzuformen. Infolge (27), (45), (56) ist

$$\left. \begin{aligned} L_{ab} &= w_{[ab}\, \pi_{2\ldots\varrho+1]} = p_{[ab,2\ldots\varrho+1]} = \frac{2}{\varrho+1} X_{[a}\, p_{b2\ldots\varrho+1]} = \\ &= \frac{4}{(\varrho+1)(\varrho+2)} X_{[a}\, \theta_{b]} + \frac{2\varrho}{(\varrho+1)(\varrho+2)} X_{[2}\, p_{3\ldots\varrho+1]ab} . \end{aligned} \right\} \; . \quad (64)$$

wo

$$\left. \begin{aligned} & p_{c_2\ldots c_\varrho\, ab} = B_{c_2}^{\lambda_1} \ldots B_{c_\varrho}^{\lambda_{\varrho-1}}\, B_a^{\lambda_\varrho}\, B_b^{\lambda_{\varrho+1}}\, (\partial_{[\lambda_1} \overset{1}{p}) \ldots (\partial_{\lambda_{\varrho+1}]} \overset{\varrho+1}{p}) ; \\ & \qquad c_2, \ldots c_\varrho = 2 \ldots \varrho+1 \quad ; \quad a, b = \varrho+2, \ldots, 2\varrho+1, j \end{aligned} \right\} \quad (65)$$

ist. (65) geht unter Berücksichtigung von (45) und (43) über in

$$\left. \begin{aligned} & p_{c_2\ldots c_\varrho\, ab} = \frac{1}{\varrho+1} (w_a\, \pi_{bc_2\ldots c_\varrho} - w_b\, \pi_{ac_2\ldots c_\varrho}) ; \\ & \qquad c_2, \ldots, c_\varrho = 2, \ldots, \varrho+1 \quad ; \quad a, b = \varrho+2, \ldots, 2\varrho+1, j \end{aligned} \right\} \quad (66)$$

und daraus geht infolge (50) hervor

$$\left. \begin{aligned} p_{c_2\ldots c_\varrho\, ab} = 0 \, (mod. \; \theta_a, \theta_b) \quad & a, b = \varrho+2, \ldots, 2\varrho+1, j \\ & c_2, \ldots, c_\varrho = 2, \ldots, \varrho+1 . \end{aligned} \right\} \; . \quad . \quad . \quad (67)$$

Wird dieses Resultat auf (64) angewandt, so ergibt sich

$$\left. \begin{aligned} L_{ab} = \frac{4}{(\varrho+1)(\varrho+2)} X_{[a}\, \theta_{b]} \, (mod. \; \theta_a, \theta_b, X_2\, \theta_a, \ldots, X_{\varrho+1}\, \theta_a, \\ X_2\, \theta_b, \ldots, X_{\varrho+1}\, \theta_b) \end{aligned} \right\} \quad (68)$$

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[4] Einfachheitshalber haben wir statt $\underset{\mathit{1}}{e}^{\varkappa}, \ldots, \underset{\mathit{n}}{e}^{x}$ und $\overset{\mathit{1}}{e}_\lambda, \ldots, \overset{\mathit{n}}{e}_\lambda$ in (61) und (62) $\underset{\mathrm{1}}{e}^{\varkappa}, \ldots, \underset{\mathrm{n}}{e}^{x}$ und $\overset{\mathrm{1}}{e}_\lambda, \ldots, \overset{\mathrm{n}}{e}_\lambda$ geschrieben.

und durch Substitution in (63) ergibt sich schliesslich

$$\left.\begin{aligned} D_j = 0\,(mod.\ \theta_{\varrho+2},\ldots,\theta_{\mathrm{p}};\ X_{[a}\theta_{b]};\ X_2\theta_a,\ldots X_{\varrho+1}\theta_a;\ X_2\theta_b,\ldots,X_{\varrho+1}\theta_b) \\ a,b=\varrho+2,\ldots,2\varrho+1,j. \end{aligned}\right\} \quad (69)$$

Die aus $\overset{1}{p},\ldots,\overset{\varrho+1}{p}$ konstruierten Grössen θ_k, $k=\varrho+2,\ldots,\mathrm{p}$ genügen schon den Gleichungen (25), die infolge (41) übergehen in

$$\left.\begin{aligned} \partial_1\theta_k = (X_2\overset{2}{\sigma}+\ldots+X_{\varrho+1}\overset{\varrho+1}{\sigma})\,\theta_k+\overset{2}{\sigma}X_2\theta_k+\ldots+\overset{\varrho+1}{\sigma}X_{\varrho+1}\theta_k - \\ -B_1^{p+1}\partial_{p+1}\theta_k-\ldots-B_1^n\partial_n\theta_k+\frac{1}{\triangle}D_k;\ k=\varrho+2,\ldots,\mathrm{p}, \end{aligned}\right\}. \quad (70)$$

wo $D_{\varrho+2}=0,\ldots,D_{2\varrho+1}=0$ zu nehmen ist. Ausserdem gilt infolge (23*b*, def.), (34) und (21)

$$\theta_k\,(\underset{0}{x^1},x^2,\ldots,x^n)=0;\ k=\varrho+2,\ldots,\mathrm{p} \quad \ldots\ldots \quad (71)$$

Nun gibt es nach CAUCHY-KOWALEWSKI nur ein System von Lösungen $\theta_{\varrho+2},\ldots,\theta_{\mathrm{p}}$ die (70) und (71) erfüllen. Offenbar ist aber $\theta_{\varrho+2}=0,\ldots,\theta_{\mathrm{p}}=0$ dieses System, denn aus (69) geht hervor, dass aus $\theta_{\varrho+2}=0,\ldots,\theta_{\mathrm{p}}=0$ das Verschwinden von D_j folgt. Damit ist aber bewiesen, dass $\overset{1}{p},\ldots\overset{\varrho+1}{p}$ auch die Gleichungen (23*b*) erfüllen. $\overset{1}{p},\ldots,\overset{\varrho+1}{p}$ erfüllen also (23*a*, *b*) und (24) und somit auch (III 4).

Der Beweis, dass $\overset{1}{p},\ldots,\overset{\varrho+1}{p}$ auch (III 5) erfüllen ist vollkommen gleichlautend mit dem in (III) gegebenen Beweis (III 36—51) für den Fall $\mathrm{p}=2\varrho+1$. Man hat nur $u-1$ durch ϱ und in (III 41) p durch $2\varrho+1$ zu ersetzen.

5. *Beweis des Teiles* (*b*) *des Haupttheorems.*

Wir gehen aus von einem Vektor Q_λ, der den Bedingungen (III 6) und (III 7) genügt. Infolge (III 3) ist dann Q_{ab} in der E_p der Indizes $1,\ldots,\mathrm{p}$ ein Bivektor vom Range 2ϱ. Diesen Bivektor zerlegen wir in ϱ Blätter und wählen in jedem Blatt einen Vektor $\overset{1}{P}_b,\ldots,\overset{\varrho}{P}_b$. Diese Vektoren sind linear unabhängig. Dazu lassen sich ϱ Vektoren $\overset{1}{P}_\lambda,\ldots,\overset{\varrho}{P}_\lambda$ konstruieren, so dass

$$\overset{h}{P}_b = B_b^\lambda\,\overset{h}{P}_\lambda;\ b=1,\ldots,\mathrm{p};\ h=1,\ldots,\varrho \quad \ldots\ldots \quad (72)$$

ist. Auch diese Vektoren sind linear unabhängig und es existiert also ein ϱ-Vektor

$$P_{\lambda_1\ldots\lambda_\varrho} \overset{\mathrm{def}}{=} \overset{1}{P}_{[\lambda_1}\ldots\overset{\varrho}{P}_{\lambda_\varrho} \quad \ldots\ldots \quad (73)$$

mit der Eigenschaft

$$P_{a_1\dots a_\varrho} \overset{\text{def}}{=} P_{\lambda_1\dots\lambda_\varrho} B^{\lambda_1}_{a_1}\dots B^{\lambda_\varrho}_{a_\varrho} \neq 0 \quad . \; . \; . \; . \; . \; . \quad (74)$$

Ausserdem folgt aus der Definition der $\overset{h}{P}_b$

$$Q_{[ab} P_{b_1\dots b_\varrho]} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (75)$$

Jetzt verfügen wir über ein Q_λ und ein $P_{\lambda_1\dots\lambda_\varrho}$, die den Bedingungen (III 6, 7, 8, 9) genügen, und nach dem schon bewiesenen Teil (a) des Haupttheorems existieren also in einer Umgebung von $x^\varkappa = \underset{0}{x}^\varkappa$ Funktionen $\overset{1}{p},\dots,\overset{\varrho+1}{p}$ die Lösungen von (III 4) sind und in $x^\varkappa = \underset{0}{x}^\varkappa$ der Gleichung (III 5) genügen. Aus (III 4) folgt aber in dieser Umgebung die Existenz von Funktionen $\underset{1}{\mu},\dots,\underset{\varrho+1}{\mu}$ sodass

$$w_\lambda \overset{\text{def}}{=} \underset{1}{\mu}\,\partial_\lambda \overset{1}{p} + \dots + \underset{\varrho+1}{\mu}\,\partial_\lambda \overset{\varrho+1}{p} \quad . \; . \; . \; . \; . \; . \; . \quad (76)$$

den Gleichungen

$$B^\lambda_b w_\lambda = 0;\; b = 1,\dots,\mathrm{p} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (77)$$

genügt und also ein Teiler von $w_{\lambda_1\dots\lambda_q}$ ist. Die μ können nicht alle zugleich verschwinden. Man kann nämlich durch umnumerieren stets erreichen dass (11) gilt und demzufolge in einer Umgebung von $x^\varkappa = \underset{0}{x}^\varkappa$ auch (24). Für $\underset{1}{\mu},\dots,\underset{\varrho+1}{\mu}$ kann man dann die ϱ-reihigen Determinanten nehmen aus der Matrix, die durch die zweite, dritte bis $(\varrho+1)$— te Reihe von (III 4) gebildet wird. Diese Determinanten können infolge (24) in $x^\varkappa = \underset{0}{x}^\varkappa$ und in einer Umgebung dieses Punktes nicht alle zugleich verschwinden. Wegen (III 5) sind ferner $\partial_\lambda \overset{1}{p},\dots,\partial_\lambda \overset{\varrho+1}{p}$ in $x^\varkappa = \underset{0}{x}^\varkappa$ und in der besagten Umgebung linear unabhängig. Es ist also dort auch $w_\lambda \neq 0$. Aus (III 5) und (76) folgt dass w_λ in $x^\varkappa = \underset{0}{x}^\varkappa$ Teiler von $Q_{[\lambda_1} P_{\lambda_2\dots\lambda_{\varrho+1}]}$ ist, sodass

$$w_\lambda = \gamma\, Q_\lambda + \underset{1}{\gamma}\,\overset{1}{P}_\lambda + \dots + \underset{\varrho}{\gamma}\,\overset{\varrho}{P}_\lambda \quad . \; . \; . \; . \; . \; . \quad (78)$$

ist, woraus durch Ueberschiebung mit B^λ_b folgt

$$0 = \underset{1}{\gamma}\,\overset{1}{P}_b + \dots + \underset{\varrho}{\gamma}\,\overset{\varrho}{P}_b \;\; ; \;\; b = 1,\dots,\mathrm{p} \quad . \; . \; . \quad (79)$$

Da aber $\overset{1}{P}_b, \ldots, \overset{\varrho}{P}_b$ linear unabhängig sind, müssen alle Koeffizienten $\underset{h}{\gamma}$ in (79) verschwinden, sodass

$$w_\lambda = \gamma\, Q_\lambda \quad \ldots\ldots\ldots\ldots \quad (80)$$

ist. Da $w_\lambda \neq 0$ ist in $x^\varkappa = \underset{0}{x}^\varkappa$, ist auch $\gamma \neq 0$ und durch geeignete Wahl des freien skalaren Faktors in $\underset{1}{\mu}, \ldots, \underset{\varrho+1}{\mu}$ kann man erreichen das $w_\lambda = Q_\lambda$ wird.

Die Klasse von w_λ ist infolge (76) höchstens gleich $2\varrho + 1$. Infolge (77) ist aber

$$2 B_b^\mu B_a^\lambda \partial_{[\mu} w_{\lambda]} = -2 (B_{[b}^\omega \partial_{|\omega|} B_{a]}^\lambda)\, w_\lambda \quad \ldots\ldots \quad (81)$$

woraus unter Berücksichtigung von (III 7) hervorgeht, dass in $x^\varkappa = \underset{0}{x}^\varkappa$

$$-2 B_b^\mu B_a^\lambda \partial_{[\mu} w_{\lambda]} = Q_{ba} \quad \ldots\ldots\ldots \quad (82)$$

ist. Nun ist der Rang von Q_{ba} gleich 2ϱ, der Rang von $\partial_{[\mu} w_{\lambda]}$ kann also in $x^\varkappa = \underset{0}{x}^\varkappa$ nicht kleiner als 2ϱ sein und zwar gilt dies, infolge (77), *für jede Wahl des freien skalaren Faktors in* w_λ. Daraus geht aber hervor, dass in $x^\varkappa = \underset{0}{x}^\varkappa$ und in der Umgebung dieses Punktes

$$w_{[\lambda}\, w_{\mu_1 \lambda_1} \ldots w_{\mu_\varrho \lambda_\varrho]} \neq 0 \quad \ldots\ldots\ldots \quad (83)$$

ist und w_λ also genau den Rang $2\varrho + 1$ hat.

6. *Beweis des Teiles* (*c*) *des Haupttheorems.*

Da der Teil (*b*) schon bewiesen ist, bleibt nur noch zu beweisen, dass es stets gelingt q linear unabhängige Vektoren $\overset{p+1}{Q}_\lambda, \ldots, \overset{n}{Q}_\lambda$ zu konstruieren, die den Bedingungen (III 6) und (III 7) genügen. Die zu diesen konstruierbaren Teiler $\overset{p+1}{w}_\lambda, \ldots, \overset{n}{w}_\lambda$ sind dann linear unabhängig, da sie in $x^\varkappa = \underset{0}{x}^\varkappa$ mit den Q_λ zusammenfallen und sie sind alle von der Klasse $2\varrho + 1$.

Wir wählen in $x^\varkappa = \underset{0}{x}^\varkappa$ das Koordinatensystem so, dass für die ersten p Massvektoren gilt

$$\underset{b}{e}^\varkappa \overset{*}{=} B_b^\varkappa;\ b = 1, \ldots, p \quad \ldots\ldots\ldots \quad (84)$$

Es seien unter den $\binom{p}{2}$ Vektoren $2 B_{[b}^\omega \partial_{|\omega|} B_{a]}^\varkappa$ gerade s linear unabhängige, die infolge der besonderen Wahl (III 10) der $B_b^\varkappa$ auch von der $B_b^\varkappa$ linear unabhängig sind. Diese führen wir in $x^\varkappa = \underset{0}{x}^\varkappa$ als s weitere Massvektoren ein

$$\underset{i}{e}^\varkappa \overset{*}{=} 2 B_{[b_i}^\omega \partial_{|\omega|} B_{a_i]}^\varkappa;\ i = p+1, \ldots, p+s \quad \ldots\ldots \quad (85)$$

Schliesslich wählen wir in $x^{\varkappa} = \underset{0}{x}^{\varkappa}$ die übrigen, von den vorherigen unabhängigen Massvektoren beliebig und schreiben diese $\underset{t}{e}^{\varkappa}$; $t = \mathrm{p} + \mathrm{s} + 1, \ldots, \mathrm{n}$. Infolge dieser besonderen Wahl ist der Ausdruck links in (III 3) für $\sigma = \varrho$ in $x^{\varkappa} = \underset{0}{x}^{\varkappa}$ von der Form

$$C_{[a_1 b_1}^{\cdots\cdots\varkappa_1} \ldots . C_{a_\varrho b_\varrho]}^{\cdots\cdots\varkappa_\varrho} = C_{\lambda_1}^{\varkappa_1} \ldots . C_{\lambda_\varrho}^{\varkappa_\varrho}\, \alpha^{i_1 \ldots i_\varrho}\, \underset{i_1}{e}^{\lambda_1} \ldots . \underset{i_\varrho}{e}^{\lambda_\varrho} \quad . \quad . \quad (86)$$

(denn $C_{\lambda}^{\varkappa} \underset{b}{e}^{\lambda} = 0$; $\varkappa = \mathrm{p} + 1, \ldots, \mathrm{n}$; $b = 1, \ldots, \mathrm{p}$ in $x^{\varkappa} = \underset{0}{x}^{\varkappa}$). Infolge (III 3) ist das rechte Glied von (86) nicht Null, und daraus folgt, dass nicht alle $\alpha^{i_1 \ldots i_\varrho}$ in $x^{\varkappa} = \underset{0}{x}^{\varkappa}$ verschwinden. Nun lässt sich (III 6) schreiben

$$Q_b = 0; \; b = 1, \ldots, \mathrm{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

und (III 7)

$$\alpha^{i_1 \ldots i_\varrho} Q_{i_1} \ldots . Q_{i_\varrho} \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

In der E_q der Indizes $i = \mathrm{p} + 1, \ldots, \mathrm{p} + \mathrm{s}$; $t = \mathrm{p} + \mathrm{s} + 1, \ldots, \mathrm{n}$ ist also der Vektor Q mit den Bestimmungszahlen Q_i, Q_t ein beliebiger Vektor, dessen E_{q-1} die Hyperfläche

$$\alpha^{i_1 \ldots i_\varrho} Q_{i_1} \ldots . Q_{i_\varrho} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

nicht berührt. Es gibt sicher q linear unabhängige Vektoren dieser Art. Wählt man dazu $Q_b = 0$, so entstehen in $x^{\varkappa} = \underset{0}{x}^{\varkappa}$ q linear unabhängige Vektoren $\overset{\varkappa}{Q}_{\lambda}$; $\varkappa = \mathrm{p} + 1, \ldots, \mathrm{n}$, die den Bedingungen (III 6) und (III 7) genügen.

Die (III, S. 454) aufgestellte Behauptung, dass es nicht möglich wäre q linear unabhängige Teiler, deren Klasse alle kleiner als $2\varrho + 1$ wären, zu konstruieren, ist unrichtig. Das PFAFFsche System

$$\left.\begin{aligned} dx^1 + x^2\,dx^3 &= 0 \\ dx^4 + x^5\,dx^6 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

bildet ein Gegenbeispiel. Der Rang 2ϱ ist 4 und dennoch gibt es zwei Teiler von Klasse 3 1). Die Existenz von q linear unabhängigen Teilern von der Klasse $2\varrho + 1$ ist also keine hinreichende Bedingung dafür, dass der Halbrang von (III 1) gleich ϱ ist. Dagegen beweist man leicht, dass der Halbrang in einem Gebiete, wo $B_b^{\varkappa}$ und $C_{\lambda}^{\varkappa}$ regulär sind, dann und nur dann gleich ϱ ist, wenn es in jedem Punkte des Gebietes Vektoren Q_{λ} gibt, die in diesem Punkte den Gleichungen (III 6) und (III 7) genügen und ausserdem jeder dieser Vektoren sich zu einem Feld w_{λ} fortsetzen lässt, das im betrachteten Gebiet Teiler von $w_{\lambda_1 \ldots \lambda_q}$ und von der Klasse $2\varrho + 1$ ist.

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5) In dem (II S. 108) als Nebenresultat erhaltenen Satz ist demnach unmittelbar nach (47*b*) das „dann und nur dann" zu streichen. Auf die weiteren Resultate hat dies keinen Einfluss.

Physiology. — *On the structure of the synapses "a distance" in the cerebellum and the retina (mossy fibres and glomeruli cerebellosi, and visual cells).* By J. BOEKE.

(Communicated at the meeting of May 25, 1940.)

Several years ago I discussed in these Proceedings (Vol. XXXII, N⁰. 6, 1929) the nature of the interneuronal connections, the synapses. In that paper I came to the conclusion, that, when we have to admit that in the synaptic region the nervous stimulus is not stopped but simply altered, polarised a.s.o., a separation by an intercellular membrane or inert intercellular material seems to be out of the question. The neuronic elements must be connected by a living substance, which is able to conduct the stimulus and connects in a certain way the neurofibrillar structures of the two elements connected. But we certainly have to account for the physiological pecularities of the synapse, and the conclusion was drawn, that we may regard the often finely striated substance of the synapse, which connects the neurofibrillar endloops of the interneuronal fibres with the internal neurofibrillar structure of the nerve cells themselves, as being of the same nature as the periterminal network of the peripheral synapses, the motor endplates and the sensory corpuscles with their protoplasmic core. This would account for their weak staining capacity and the difficulty of demonstrating the striated structure of the interneuronal synapses in a satisfactory manner. It was pointed out by me that if we could regard these interneuronal junctions as of the same nature as the periterminal network of the different peripheral nerve-endings, the physiologic independance of the neurones in relation to drug-action and function, and the peculiar way in which the synapse differs from both the nerve cell and the terminal branches of the nerve fibres, would be accounted for in a satisfactory way, together with the anatomical continuity of structure (and connection by living substance) between them which we have to acknowledge in the light of modern histology (see BOEKE, Problems of Nervous Anatomy, Oxford University Press, 1940).

In the synapse there must be a connection of the two neuronic parts by *living* substance. This is the only way to account for the transmission of the nervous stimulus and its alterations. Even the hypothetical synaptic membrane of SHERRINGTON must be an arrangement of units of the *living* substance, and this arrangement may be present in the periterminal network, not as a real visible membrane, but as a biphasic condition of the living substance itself. If we regard the intervening synaptic substance as a sort of inert, non-living cement substance, sometimes faintly striated, as the classical neuronists do, a connection in which no alteration is possible, we remain in the dark and no real histological basis for the modern conceptions of the function of the nervous system and the nature of the transmission of the nervous stimulus is possible. We have to regard the synaptic substance as a living substance, in which there is present a structure akin to the periterminal network of the peripheral junctions and connecting the neurofibrillar endloops with the neurofibrillar network inside the cell. In this way the synapse may alter its structure as living substance does, it may obtain a more definite linear arrangement of the units composing its connecting fibrils, inside its protoplasmic substance there may be formed a more complex organisation of its linear structure, a transient part of this linear structure may become fixed, in short, inside the synaptic *living* substance the conducting apparatus may become more and better organised (learning, "Bahnung" etc.). Even the structure of the sarkoplasmic base of the motor endplate with its periterminal network may be altered according to its functional condition (cf. BOEKE, l.c. 1940).

In most cases the structure of synaptic junctions nowadays is regarded as consisting of living substance ("protoplasmic continuity is, therefore, the rule at synaptic junctions", GERARD, 1934, in BLOOMs Textbook).

The most difficult case to account for is the synaptic junction in the so-called glomeruli olfactorii and the glomeruli cerebellosi, in which the terminal arborisations of the axons and of the dendrites of the nervous elements of the two different neurons form a very dense entanglement with no evidence whatever of any protoplasmic continuity between them. Indeed, with the exception of HELD (1897) and PENSA (1931) every author follows the footsteps of CAJAL, who described them in 1926 as a mass of inert substance, which presented no definite structure whatever, and in which the telodendria of the different nerve fibres branched and intertwined without coming into contact with each other. After CAJAL, HORNE CRAIGIE (1926), LUGARO, JACOB (1928) and many others the glomeruli form an independant mass of substance (JACOB calls them "Protoplasmainsel") with a sharp outline, in which the nerve branches enter after having lost their myelin sheath. PENSA describes in the glomeruli cerebellosi special nuclei, belonging to oligodendroglia cells, and thus maintains that the groundsubstance is nothing but glious tissue.

According to HELD (1897) the glomeruli cerebellosi form a syncytial mass of protoplasm ("Grundplasma") with neurosomes and in close syncytial connexion with the protoplasm of the surrounding nerve cells. The neurosomes are continuous with the neurosomes of the nerve telodendria entering the glomeruli.

In order to investigate the way in which in these synaptic junctions the connexions between the different nerve fibres and their conducting structure are established, I studied the glomeruli in the cortex of the cerebellum of the cat and of the rabbit and the retina of the horse and of the cat, in silver preparations and after having fixed the tissue as carefully as possible with corrosive sublimate, Hermanns fluid and a mixture of both.

Three questions have to be studied:

a. what is the nature of the mass of the glomeruli? Is it inert material or living protoplasmatic substance? Is it of a glious nature or of nervous origin? Are nuclei present inside the glomerulus?

b. Is this groundsubstance of the glomeruli independant of the surrounding tissue or not?

c. In what manner are the nerve branches entering the glomerulus connected with each other, is there any evidence of a protoplasmic continuity between them?

ad a. In the first place it is interesting to note that in the glomeruli there is a very high metabolic activity, which points to be supposition, that their substance is *living* substance. The glomeruli are surrounded by a close network of capillaries, and everywhere in the brain, wherever synaptic junctions of this type are abundant, the capillaries form a closer mesh than elsewhere. This suggests that these synapses are a region of high metabolic activity, and the same suggestion is made by NOYONS, who found in his wonderfully exact physiological experiments, that the basal metabolism of the cortex of the cerebellum with its extremely numerous synapses is far greater than of the cortex cerebri, where the synaptic junctions are less numerous, much smaller and never form large masses of groundsubstance as in the glomeruli of the cortex cerebelli.

This high metabolism needs a living groundsubstance and not an inert material. Thus JACOB is perfectly right when he calls the glomeruli cerebellosi "Protoplasmainsel", and it is curious to note in this connection, that CAJAL himself has felt the difficulty to regard them simply as consisting of inert cement substance as the neurone theory required. In his drawings as in those of his collaborators (f.i. HORNE CRAIGIE) they appear as sharply defined homogeneous grey spots in which the nerve branches intertwine, but even in 1926 CAJAL in the text compares their substance with the groundsubstance of the motor endplates, which must be able to conduct the nervous stimulus from the terminal arborisation of the motor nerve to the contractile substance of the striated muscle fibre. Now every histologist without any exception regards the groundsubstance of the sole plate as protoplasm, sarcoplasma, as *living substance*, and I may quote here GERARD (in BLOOMS Textbook, 1934): "in the motor end-plate there is obviously protoplasmic continuity between nerve fibre and muscle fibre; but the protoplasm of the end-plate is visibly different from that of both nerve and striated muscle. The physiologic properties of this "junctional tissue" are also unique." GERARD then bases his account of the peculiar physiologic features of synaptic junctions in part upon these motor end-plates "where the junctional tissue is visible" In this protoplasmic junctional tissue of the motor end-plates the periterminal network is described, and even CAJAL himself admits that this periterminal network is of protoplasmic origin and may possibly form a link in the conducting process of the nervous stimulus (CAJAL, 1934). Now, as will be described farther on, the same periterminal network is to be seen in the groundsubstance of the glomeruli, and so we may be convinced that it is of protoplasmic origin and that it is living substance.

The question whether it is of glious nature as was maintained by PENSA (1931), or of nervous nature as was maintained by HELD (1897) is difficult to answer. I have never been able to find the nuclei and the oligodendroglia cells PENSA describes inside the glomeruli. Glious cells and nuclei are to be found everywhere in the granular layer of the cortex cerebelli, but I have never found a single nucleus inside the groundsubstance of the glomeruli. Neurosomes are to be found inside this groundsubstance and when we study carefully-made thin sections we may see, that these granules are in continuous connexion with the granules inside the terminal branches of the nerve fibres and their neurofibrillar expansions. The substance of the glomeruli is everywhere in continuous connexion with the neuroplasm of the entering nerve branches; they lose their myelin sheath (when they are myelinated) as soon as they have entered the substance of the glomerulus, and not before. In my opinion this groundsubstance is of an intermediate character and of protoplasmic origin.

ad b. The groundsubstance of the glomeruli is never independant of the surrounding tissue as it appears in the drawings of CAJAL, HORNE CRAIGIE and other neuronists. It is a part of this tissue and it is everywhere in continuous protoplasmic connexion with the surrounding elements, the granular cells and the mossy fibres. This connexion is only possible when we admit that the groundsubstance is living protoplasm and that it is of nervous origin (and intermediate physiologic character).

ad c. Inside the groundsubstance of the glomeruli a distinct netlike structure, a periterminal network may be made visible, and this network is everywhere in continuous connexion with the neurofibrillar structure of the terminal arborisations of the entering nerve fibres. In figure 1 I have drawn as accurately as possible what my preparations showed me. The network is very delicate, regular with small meshes, exactly as it is described for the periterminal network of the motor end-plates and the sensory corpuscles (BOEKE, Zeitschr. Mikr.-Anat. Forschung, Bd. IV, 1926). It is attached everywhere to the terminal neurofibrillar branches of the nerve fibres and extends throughout the whole of the glomerulus, fading away at its border. The nerve fibres are therefore in evident protoplasmic continuity, but the protoplasm of the glomerulus has an intermediate character as it was stated for the sarcoplasm of the sole-plate with its periterminal network. It conducts the nervous stimulus and appears to possess all the characters of a real synaptic junction.

It is not possible to describe all its features and to discuss them in this short preliminary communication. I hope to give a better and more

J. BOEKE: ON THE STRUCTURE OF THE SYNAPSES "A DISTANCE" IN THE CEREBELLUM AND THE RETINA (MOSSY FIBRES AND GLOMERULI CEREBELLOSI, AND VISUAL CELLS).

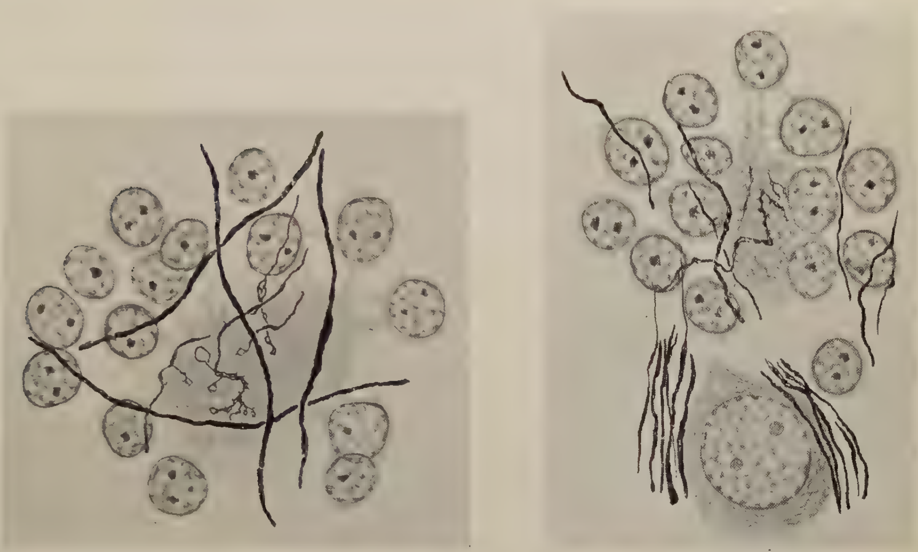

Fig. 1. Glomeruli cerebellosi in the granular layer of the cortex cerebelli of the rabbit with periterminal network and terminal neurofibrillar nerve branches. To the right below a Purkinje-cell.

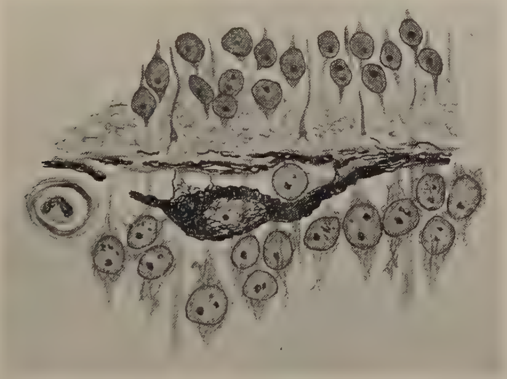

Fig. 2. Cross-section through the retina of a cat in the region between the ends of the visual cells and the cells of the second neuron. The black element is a horizontal cell.

Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

detailed description in the XI. Innervationsstudie in the Zeitschr. f. Mikr.-Anat. Forschung, with more figures and a discussion of the details.

I may add to this description that exactly the same connexions are to be found in the retina between the endknobs of the visual cells and the terminal branches of the second neuron, as it is drawn in fig. 2. Here too we find distinct protoplasmic connexions as it was maintained allready by AKKERINGA (1936) for the ganglion cells of the second and third neuron.

Utrecht, May 1940.

Psychology. — *Occipital forms, facial dimensions, age, intellect and character.* By E. D. WIERSMA.

(Communicated at the meeting of May 25, 1940.)

The investigations on "Facial dimensions, age and intellect" [1]) demonstrated that a connection exists between the dimensions of the face, the age, and the intellectual powers. The following investigation was made with the object of ascertaining in how far the dissimilitude in the form of the occiput is coupled with differences in psychical phenomena (temperament, intellect and character), in bodily build and in physiological functions. The said dissimilitude in the occipital forms concern the prominence or sharp regression of the back of the head. They are frequently distinguishable by the eye. In doubtful cases the distance is measured from the glabella to the vertex, to the middle of the occipital curve and to the limit of the hair growth at the back of the neck. In respect of the prominent occiput, these measurements show greater differences, and in respect of the receding occiput, lesser differences. The material for this investigation was afforded me in various ways.

1. Dr. L. KAISER took photos en profile of 217 pupils (117 male and 100 female ones). In many of these it can be plainly seen whether the occiput is prominent or steeply receding, and in those of the male students it can further be seen whether the prominence is most pronounced in the upper part, the middle part, or lower part of the head.

Not all the photos could be made use of for this investigation, because the hair, especially of the female students, does not allow of the exact form of the back of the head being distinguished in the photos. As Dr. KAISER comes into close contact with her pupils, she was able, in the majority of cases to state the psychical qualities with regard to temperament, intellect and character. I am greatly indebted to Dr. KAISER for this material.

2. In the somato-psychological enquête [2]), I inquired, in question 115, whether the occiput of the individual in question projected or receded. These forms were selected from the enquête material and in both cases it was investigated what differences there were in bodily and mental qualities.

3. In the above-mentioned enquête there was further inquired whether

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[1]) Proc. Kon. Ned. Akad. van Wetensch., Amsterdam, 42, 727 (1939).

[2]) Capita psychopathologica, p. 373—377.
Ter overdenking voor huisarts en student, p. 42, 46—75.
Voor paedagoog en huisarts, p. 56 and following.

the individual in question was shortheaded or longheaded. It is known that in longheaded persons a very large percentage of individuals exhibit a prominet occiput, whereas this is strongly receding in the case of shortheaded persons. The Table below will show this.

Figures of the percentages.

	Longheaded	Shortheaded
Prominent occiput	62.6	17.1
Receding	26.8	42.2

4. The said enquête further supplies information respecting differences in psychical qualities of the fine and slender persons and the thicksets, and teaches that persons with a prominent occiput are more fine and slender and less thickset, while the persons with receding occiput are less fine and slender and more thickset than the average.

Percentage figures.

	Prominent occiput	Receding occiput	Average
Fine and slender persons	43.2	32.8	38.1
Thicksets	9.5	12.4	10.6

5. Measurements were also taken of mentally normal and abnormal persons of widely different ages in the Children's Hospital, in the Surgical Clinic, in the Open Air School, and in the schools for feeble-minded and for imbecile children, at Groningen. My thanks are due to the Heads of these Institutes for permission to enter their institutions, for the assistance afforded me, and for the information supplied. Compasses-measurements ware taken of the distance from the glabella tot the vertex, from glabella to the middle of the occipital curve, from the glabella to the limit of the hair-growth in the neck, and, further, the length of the forehead, of the upper and of the lower jaw, as has been indicated elsewhere [1]).

Below I give the investigation of the above-mentioned groups.

In order to judge of the differences in intellect in her material, Dr. KAISER (group 1) attached value to the existence of special gifts, to adroitness, succinctness and matter of factness in their narration. The enquête material (groups 2, 3 and 4) was judged from the answers to questions as to quickness of grasp and good sense, or as to superficiality and stupidity,

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[1]) Proc. Kon. Ned. Akad. van Wetensch., Amsterdam, 42, 727 (1939).

Temperamental Qualities.

	Group 1		Group 2			Group 3[1]			Group 4[2]	
	A	n A	A	n A		A	n A		A	n A
Average	57.3	42.7	63.7	37.1	Average	66.7	23.2	Average	66.6	23.4
Prominent occiputs	54.2	45.7	60.5	37.1	Longheaded	63.6	27.1	Fine and slender persons	65.6	25.3
Receding occiputs	63.3	36.7	67.0	32.9	Shortheaded	66.9	20.8	Thicksets	73.0	20.5
	E	n E	E	n E		E	n E		E	n E
Average	61.5	38.5	62.9	37.1	Average	65.3	16.9	Average	65.3	16.9
Prominent occiputs	68.1	31.9	69.1	30.8	Longheaded	64.9	19.2	Fine and slender persons	68.9	18.0
Receding occiputs	46.7	53.3	56.9	43.3	Shortheaded	69.4	16.2	Thicksets	57.1	14.3
	S	P	S	P		S	P		S	P
Average	38.4	61.6	31.9	68.2	Average	40.3	26.6	Average	46.5	24.1
Prominent occiputs	30.6	69.4	33.3	66.6	Longheaded	41.8	25.6	Fine and slender persons	46.9	24.6
Receding occiputs	56.7	43.3	53.2	44.5	Shortheaded	38.3	31.1	Thicksets	53.2	18.8

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[1] Zeitschr. f. angewandte Psychologie. B. **38**, p. 151.

[2] Zeitschr. f. angewandte Psychologie. B. **38**, p. 162.

to more or less knowledge of human nature, to broad-mindedness or narrow-mindedness, to individuality or an inclination to imitation, to the greater or lesser powers of observation. Intellectual differences between the persons of group 5 can, for a part, be inferred from the schools which they attend, and were further supplied me by the Heads of the Institutions, which, in many cases, have the I. Q. of the pupils at their disposition.

In forming a judgment of the character the most important point is to ascertain whether the individual in question is chiefly preoccupied with himself, or has always interest in, and a fellow-feeling for, others; whether he thinks in the first place of his own interests or is in the habit of entering into the difficulties of the persons in his environment; in how far thus, egoistic or altruistic motives dominate in his thoughts and actions. Egoism is stamped upon all the lower character traits, on the vital, egophile and the lower abstract tendencies.

For the reasons above-mentioned, in the investigation of group I, I made use only of the photos of the male students. (See table on page 694).

In these four groups persons with a receding occiput possess a greater activity, and persons with a prominent occiput less, than the average. This similarity in the groups exhibits a few exceptions in regard to the emotionality and the psychical after-effects. From the Table below, in which the average values of all the groups are given, it appears that persons with a very receding occiput are more active, less emotional, and of secondary importance than the average, and than the persons with a prominent occiput.

Average	A	n A	E	n E	S	P
Of all occiputs	62.2	31.4	63.7	27.3	46.8	45.1
Of prominent occiputs	60.9	33.8	67.7	24.9	38.1	46.6
Of receding occiputs	67.5	27.7	57.5	31.8	50.3	34.4

Intellect and character.

Intellect and character are in a high degree dependent on the temperament. It is known that mental grasp as well as superficiality is promoted by primary functions; good sense and broad-mindedness, as well as stupidity, by secondary functions; violence and irritability, as well as compassion and helpfulness, by the sensibilities; perseverance and thoughtfulness, as well as suspiciousness and distrust, by the secondary functions; receptivity of new ideas, as well as contradictions in thought and action, by the primary functions. On the grounds of these differences found in all the groups, a survey of the average values is given, as in the temperamental qualities.

In judging the intellect of the persons in the several groups, it was noted whether the above-enumerated symptoms ware evinced or not.

	Group 1		Group 2	
	Intellectual	Not intellectual	Intellectual	Not intellectual
Avarage	63.9	4.2	59.6	9.9
Prominent occiput	62.1	2.5	62.3	9.4
Receding occiput	50.0	3.3	58.1	10.5

The investigation of the longheaded and shortheaded, the fine and slender persons and thicksets, indicates that the longheaded are more secondary, the shortheaded more primary than the average (see Table of the temperamental qualities). In accordance with this, the longheaded are less receptive and more understanding than the shortheaded. The thicksets are more secondary than the fine and slender persons and, accordingly, more understanding.

In the same way the persons in groups 2, 3 and 4 are distinguished in egoistic and altruistic thinking and feeling individuals. The degree of their egoism is evinced in their more ore less indulgence in the satisfying of vital tendencies, in the lessened altruism, their feebler sense of duty and love of truth, their greater suspiciousness and unreliability.

	Group 1		Group 2	
	Egoism	Altr.	Egoism	Altr.
Average	29.9	23.0	18.7	55.1
Prominent occiputs	30.5	20.9	20.1	55.4
Receding occiputs	20.0	30.0	19.7	56.8

	Group 3			Group 4	
	Egoism	Altr.		Egoism	Altr.
Average	22.4	65.7	Average	30.8	47.6
Longheaded	21.6	65.3	Fine and slender	29.3	48.6
Shortheaded	25.7	63.2	Thicksets	27.4	51.7

From the above Table it is apparent that the persons with a very receding occiput stand on a higher moral plane than the persons with a prominent occiput. This holds good for all the groups, except Group 3.

Group 5. The measurements taken in the above-mentioned institutions of 205 persons are expressed in the Table below in percentages of the middle skull-line.

Influence of Age

Age	Number	Upper line	Lowest line	Height of forehead	Length of upper jaw	Length of lower jaw	Length of jaw system
0— 1 y.	21	95.3	97.7	26.1	36.5	15.6	52.1
1— 2 y.	20	97.8	91.2	26.2	36.9	17.0	53.9
2— 5 y.	16	93.8	90.9	29.6	38.3	17.9	56.2
5—12 y.	46	97.3	90.0	26.3	44.1	18.0	62.1
12—18 y.	15	97.7	94.0	28.4	43.1	19.5	62.6
above 18	87	96.2	95.8	29.8	44.4	22.5	66.9

From the above Table it will be seen that the height of the forehead, the length of the upper jaw, of the lower jaw, and of the whole jaw system becomes regularly greater with increasing age in proportion to the middle line of the skull.

Influence of Sex.
Of the 87 adults measured, 47 were males and 40 females.

Sex	Number	Upper skull	Lower skull line	Length of forehead	Length of upper jaw	Length of lower jaw	Length of jaw system
Male	47	96.3	95.8	30.7	44.3	22.1	66.4
Female	40	96.0	95.8	28.9	44.7	21.0	65.7
Male and Female	87	96.2	95.8	29.8	44.4	21.5	65.9

The differences in facial and skull dimensions is of little significance.

Influence of intellectual gifts.

Of 92 pupils in the school for feeble-minded children the I. Q. fluctuated between 60 and 90, of 72 pupils in the school for imbecile children the I. Q. was lower than 60. These feeble-minded and imbeciles are compared in the following Table with normal individuals of about the same age (5—18).

Number	Mental cond.	Upper skull line	Lower skull line	Length of forehead	Length upper jaw	Length lower jaw	Length of jaw system
61	normal	97.5	92.0	27.2	43.6	18.9	62.5
92	feeble-minded	96.0	90.5	23.4	43.8	20.7	64.4
72	imbecile	98.9	86.2	23.6	48.9	19.0	67.9

It will be evident from the above Table that in the imbeciles the upper line of the skull differs most from the middle, the occiput projecting most in the middle. With the normal, the forehead is higher, the upper and the lower jaw and the whole jaw system smaller, in proportion to the middle skull-line than with the feeble-minded and the imbeciles.

In the Open-air School, where children with various bodily infirmities are taught, measurements were taken of 28 children. According to the information from the resident nurse and the teacher, 16 children are intellectually normal and 12 are feeble-minded. The following Table gives a survey of the measurements.

Number	Mental cond.	Upper skull line	Lower skull line	Length of forehead	Length of upper jaw	Length of lower jaw	Length of jaw system
12	feeble-minded	99.4	89.8	24.5	45.0	16.9	
16	normal	96.2	87.8	26.2	43.3	19.4	

This Table, too affords evidence that the forehead of the feeble-minded is smaller than normal size, while the upper jaw is larger than the normal.

Physics. — *The decay of the penetrating cosmic rays.* IV. By E. M. BRUINS. (Communicated by Prof. J. CLAY).

(Communicated at the meeting of May 25, 1940.)

In order to explain the different absorption of penetrating cosmic rays in air and in condensed materials FERMI[1]) suggested that the energyloss of the particles in a condensed material is less then in air. In the present paper[2]) we will compare this hypothesis with that of the decaying electrons. The energyspectrum, being a homogeneous function of degree s, no information can be obtained by measuring the distribution of intensity around the vertical, for in *both* cases this has to be

$$I(\vartheta) = I_0 \cos \vartheta^{s-1}.$$

At sealevel, only from a comparison of measurements in one *fixed* direction with different layers of absorbing materials with that in another direction *and* with measurements under the same shield in different directions information about this question can be obtained.[3])

Now the measurements of the variation with zenith-angle ϑ confirmed, in a complete accord with the integral energyspectrum $E^{-1.93} \simeq E^{-2}$ found at greater depths below the sealevel, the $\cos^2 \vartheta$-distribution.

The main difference between the decaying electrons (a) and the FERMI-hypothesis (b) lies in the fact that in different directions the energy distribution remains *unchanged* in the latter case whereas *the decay changes the energyspectrum* as has been discussed in a former paper (Decay I).

If t represents the energyloss expressed in the loss in the atmosphere as a unit we have

a. Decay: $I(\vartheta, t=1) = I_0\, C(1) \cos \vartheta^{s-1} = I(\vartheta)$
$I(0, t) = I_0\, C(t)\, t^{1-s} = I(t)$

b. FERMI: $I(\vartheta, t=1) = I_0 \cos \vartheta^{s-1} = I(\vartheta)$
$I(0, t) = I_0\, t^{1-s} = I(t)$

where $C(t)$ is the correctionfactor due to the decay which has been calculated by numerical integration (Decay II, III).

a. Comparing the intensity under layers of equal mass per cm^2 we obtain, when a represents the energyloss per gr. per cm^2 of air and m the same quantity per gr. per cm² of matter, y, z being the massequivalents of the layers of air and x that of the layer of absorbing material:

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[1]) E. FERMI, Phys. Rev. 56, 1242 (1939).

[2]) Comp. Decay I, II, III, Proc. Kon. Ned. Akad. van Wetensch., Amsterdam, **42**, 54, (1939); **42**, 740, (1939); **43**, 75, (1940).

[3]) Comp. J. CLAY. Proc. Kon. Ned. Akad. van Wetensch., Amsterdam, **43**, 437—438, (1940).

a. Decay: $$\frac{I(\vartheta)}{I(t)} = \frac{C(1)}{C(t)} \quad \text{or} \quad \lim_{t \to \infty} \frac{I(\vartheta)}{I(t)} = C(1).$$

The limiting value $1 : C(1) = I(t) : I(\vartheta)$ is given in table I and shown in fig. 1 as a function of the lifetime τ.

b. FERMI: $$\frac{I(\vartheta)}{I(t)} = \left(\frac{ay + mx}{az}\right)^{s-1}$$

or with $m = a - f$ as $z = x + y$

$$\frac{f}{a} = 1 - \sqrt[s-1]{\frac{I(\vartheta)}{I(t)}}.$$

TABLE I.

τ	1	2	4	8	10	$\times 10^{-6}$
$I(t)\ I(\vartheta)$	8.00	3.50	2.03	1.49	1.39	

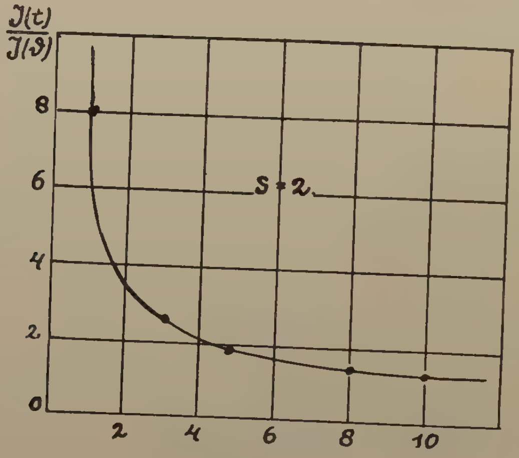

Fig. 1.

β. The relation between ϑ and t for which both intensities reach the same value is

a. Decay: $$\underline{\sec\vartheta = \sqrt[s-1]{\frac{C(1)}{C(t)}} \cdot t}$$

b. FERMI: $$mx + ay = az = ay \sec\vartheta \quad \text{or}$$

$$\underline{(\sec\vartheta - 1) = \frac{m}{a}(t - 1).}$$

For non decaying particles the *sec* ϑ-t-diagram gives a *straight line* the slope of which is determined by $\frac{m}{a} < 1$.

For decaying particles this diagram shows a curve, the slope of which approaches $\sqrt[s-1]{C(1)}$ for great values of t whereas the initial slope can be calculated as $1-\frac{1}{2}\left(\frac{dl_n C(t)}{dt}\right)_{t=1}$

These values are given in table II and the curves for different lifetime τ are shown in fig. 2.

TABLE II.

$\tau =$	1	2	4	8	10	∞	$x\,10^{-6}$
limes slope: $t \rightarrow \infty$	0.354	0.535	0.702	0.821	0.848	1	
initial slope $t=1$	0	0.184	0.427	0.578	0.628	1	

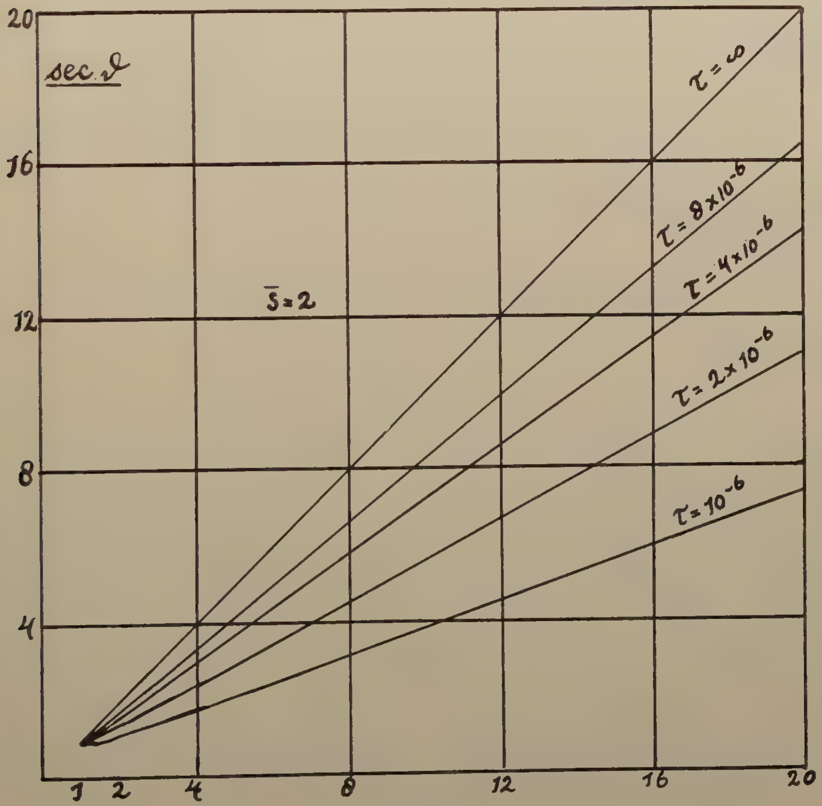

Fig. 2.

A deviation from the straight line through *sec* $\vartheta = 1$, $t = 1$ is in *contradiction* with the hypothesis of FERMI.

Mathematics. — *Ueber eine Erweiterung der* LAPLACE-*Transformation.* (Zweite Mitteilung). Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of May 25, 1940.)

Hilfssatz 4. *Ist* $0 < \delta < x$ *und* $\beta > 0$, *so gilt*

$$\lim_{\lambda \to \infty} \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, s\, ds \int\limits_{x}^{x+\delta} K_\nu(st)\, t^{\nu+1}\, dt = \tfrac{1}{2}\pi i x^\nu \quad . \quad . \quad . \quad . \quad (26)$$

und

$$\lim_{\lambda \to \infty} \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, s\, ds \int\limits_{x-\delta}^{x} K_\nu(st)\, t^{\nu+1}\, dt = \tfrac{1}{2}\pi i x^\nu. \quad . \quad . \quad . \quad (27)$$

Beweis. Bekanntlich ist [18])

$$\int K_\nu(z)\, z^{\nu+1}\, dz = -z^{\nu+1} K_{\nu+1}(z) + C. \quad . \quad . \quad . \quad . \quad (28)$$

Folglich hat man

$$\lim_{\lambda \to \infty} \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, s\, ds \int\limits_{a}^{b} K_\nu(st)\, t^{\nu+1}\, dt$$

$$= a^{\nu+1} \lim_{\lambda \to \infty} \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, K_{\nu+1}(as)\, ds - b^{\nu+1} \lim_{\lambda \to \infty} \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, K_{\nu+1}(bs)\, ds.$$

Die beiden rechtstehenden Integrale können mit Hilfe von (23) und (24) berechnet werden. Nimmt man $a = x$ und $b = x + \delta$, so bekommt man (26); für $a = x - \delta$ und $b = x$ findet man (27).

Hilfssatz 5. *Ist* $0 < \alpha < d$ *und* $\mu \geqq 0$, *so hat man*

$$\left| \int\limits_{\alpha-\mu i}^{\alpha+\mu i} \frac{e^{-\zeta}\, d\zeta}{\zeta} \right| < M, \qquad \left| \int\limits_{\alpha-\mu i}^{\alpha+\mu i} \frac{e^{\zeta}\, d\zeta}{\zeta} \right| < M,$$

wo M *nicht von* α *und* μ, *also nur von* d *abhängt.*

---

[18]) WATSON, [22], 79, Formel (3).

Beweis. Für $a > 0$ gilt bekanntlich

$$\int_{\alpha-\infty i}^{\alpha+\infty i} \frac{e^{-\zeta}\,d\zeta}{\zeta} = 0, \qquad \int_{\alpha-\infty i}^{\alpha+\infty i} \frac{e^{\zeta}\,d\zeta}{\zeta} = 2\pi i. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Ferner hat man [19])

$$\int_{\alpha+\mu i}^{\alpha+\infty i} \frac{e^{-\zeta}\,d\zeta}{\zeta} = G(a+\mu i), \qquad \int_{\alpha-\mu i}^{\alpha-\infty i} \frac{e^{-\zeta}\,d\zeta}{\zeta} = G(a-\mu i), \quad . \quad (30)$$

$$\int_{\alpha+\mu i}^{\alpha+\infty i} \frac{e^{\zeta}\,d\zeta}{\zeta} = G(-a-\mu i), \quad \int_{\alpha-\mu i}^{\alpha-\infty i} \frac{e^{\zeta}\,d\zeta}{\zeta} = G(-a+\mu i); \quad . \quad . \quad (31)$$

hierin ist

$$G(z) = -\gamma - \log z - \sum_{r=1}^{\infty} \frac{(-z)^r}{r\,\Gamma(r+1)} = e^{-\frac{1}{2}z}\, z^{-\frac{1}{2}}\, W_{-\frac{1}{2},0}(z), \quad . \quad (32)$$

wo γ die EULERsche Konstante und $W_{k,m}(z)$ die WHITTAKERsche Funktion bezeichnet.

Aus (29), (30) und (31) ergibt sich nun

$$\int_{\alpha-\mu i}^{\alpha+\mu i} \frac{e^{-\zeta}\,d\zeta}{\zeta} = G(a-\mu i) - G(a+\mu i), \quad . \quad . \quad . \quad . \quad (33)$$

$$\int_{\alpha-\mu i}^{\alpha+\mu i} \frac{e^{\zeta}\,d\zeta}{\zeta} = 2\pi i + G(-a+\mu i) - G(-a-\mu i). \quad . \quad . \quad (34)$$

Für die Funktion $G(z)$ gilt aber [20]) für festes $\varepsilon > 0$ und für $|z| \to \infty$

$$G(z) = e^{-z} z^{-1} \{1 + O(z^{-1})\} \qquad (-\tfrac{3}{2}\pi + \varepsilon < \arg z < \tfrac{3}{2}\pi - \varepsilon). \quad (35)$$

Die Behauptung des Hilfssatzes ergibt sich jetzt wegen (33) und (34) sofort aus (32) und (35), da $\left|\log\left(\frac{a+\mu i}{a-\mu i}\right)\right| < \pi$ und $\left|\log\left(\frac{-a-\mu i}{-a+\mu i}\right)\right| < 2\pi$ ist.

Hilfssatz 6. *Ist* $0 < c < a < d$ *und* $\mu \geqq 0$, *so hat man*

$$\left|\int_{\alpha-\mu i}^{\alpha} \frac{e^{-\zeta}\,d\zeta}{\zeta}\right| < N, \qquad \left|\int_{\alpha}^{\alpha+\mu i} \frac{e^{-\zeta}\,d\zeta}{\zeta}\right| < N,$$

wo N *nicht von* a *und* μ, *also nur von* c *und* d *abhängt.*

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[19]) Man vergl. GOLDSTEIN, [6], 112.

[20]) Man vergl. WHITTAKER and WATSON, [23], § 16.3—16.4.

Beweis. Die Integrale sind stetige Funktionen von α und μ, die bei unbeschränkt wachsendem μ nach einem endlichen Grenzwert streben.

§ 3. Beweis von Satz 1.

Ich zerlege den Beweis in drei Teile[21].

Erster Teil. Für hinreichend kleine positive Werte von δ $(0<\delta<x)$ gilt

$$\frac{1}{\pi i}\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,(xs)^{\frac{1}{2}}\,ds\int_{x}^{x+\delta} K_\nu(st)\,(st)^{\frac{1}{2}}\,F(t)\,dt=\tfrac{1}{2}\,F(x+0) \qquad (36)$$

und

$$\frac{1}{\pi i}\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,(xs)^{\frac{1}{2}}\,ds\int_{x-\delta}^{x} K_\nu(st)\,(st)^{\frac{1}{2}}\,F(t)\,dt=\tfrac{1}{2}\,F(x-0). \qquad (37)$$

Zweiter Teil. Ist $0<\delta<x$, so gilt

$$\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,(xs)^{\frac{1}{2}}\,ds\int_{0}^{x-\delta} K_\nu(st)\,(st)^{\frac{1}{2}}\,F(t)\,dt=0. \quad . \quad . \quad . \qquad (38)$$

Dritter Teil. Ist $\delta>0$, so gilt

$$\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,(xs)^{\frac{1}{2}}\,ds\int_{x+\delta}^{\infty} K_\nu(st)\,(st)^{\frac{1}{2}}\,F(t)\,dt=0. \quad . \quad . \quad . \qquad (39)$$

Formel (5) folgt nun sofort aus (36), (37), (38) und (39).

Beweis von (36) und (37). Die Funktion $F(t)$, also auch $t^{-\nu-\frac{1}{2}}F(t)$, ist von beschränkter Variation in der Umgebung des Punktes $t=x$. Für hinreichend kleine positive Werte von τ hat man also im Intervall $x<t\leqq x+\tau$

$$t^{-\nu-\frac{1}{2}}F(t)=x^{-\nu-\frac{1}{2}}F(x+0)+\varphi_1(t)-\varphi_2(t); \quad . \quad . \quad . \quad . \qquad (40)$$

hierin sind $\varphi_1(t)$ und $\varphi_2(t)$ monoton wachsende Funktionen mit $\lim\limits_{t\to x}\varphi_1(t)=0$, $\lim\limits_{t\to x}\varphi_2(t)=0$.

Bei hinreichend kleinem τ gilt analog im Intervall $x-\tau\leqq t<x$

$$t^{-\nu-\frac{1}{2}}F(t)=x^{-\nu-\frac{1}{2}}F(x-0)+\psi_1(t)-\psi_2(t), \quad . \quad . \quad . \quad . \qquad (41)$$

---

[21] Der Beweis von Satz 1 ist mit dem TITCHMARSHschen Beweis der HANKELschen Formel (4) verwandt; man vergl. TITCHMARSH, [21], 240—242.

wo $\psi_1(t)$ und $\psi_2(t)$ monoton abnehmende Funktionen mit $\lim\limits_{t \to x} \psi_1(t) = 0$, $\lim\limits_{t \to x} \psi_2(t) = 0$ bedeuten.

Zu jedem $\varepsilon > 0$ kann man nun ein δ $(0 < \delta \leqq \tau)$ so wählen, dass

$$\varphi_1(x+\delta) < \varepsilon,\ \varphi_2(x+\delta) < \varepsilon,\ \psi_1(x-\delta) < \varepsilon,\ \psi_2(x-\delta) < \varepsilon \quad . \quad (42)$$

ist.

Nun hat man mit Rücksicht auf (40)

$$\left.\begin{aligned} &\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)(xs)^{\frac{1}{2}}\, ds \int\limits_{x}^{x+\delta} K_\nu(st)(st)^{\frac{1}{2}} F(t)\, dt \\ &\qquad = x^{-\nu} F(x+0) \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, s\, ds \int\limits_{x}^{x+\delta} K_\nu(st)\, t^{\nu+1}\, dt \\ &\qquad + x^{\frac{1}{2}} \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, s\, ds \int\limits_{x}^{x+\delta} K_\nu(st) \left\{\varphi_1(t) - \varphi_2(t)\right\} t^{\nu+1}\, dt. \end{aligned}\right\} \quad . \quad . \quad (43)$$

Das Integral mit $\varphi_1(t)$ lässt sich unter Anwendung des zweiten Mittelwertsatzes folgenderweise reduzieren 22):

$$\left.\begin{aligned} & x^{\frac{1}{2}} \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, s\, ds \int\limits_{x}^{x+\delta} K_\nu(st)\, \varphi_1(t)\, t^{\nu+1}\, dt \\ & = x^{\frac{1}{2}} \int\limits_{x}^{x+\delta} \varphi_1(t)\, t^{\nu+1}\, dt \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs) K_\nu(st)\, s\, ds \\ & = x^{\frac{1}{2}} \varphi_1(x+\delta) \int\limits_{\eta}^{x+\delta} t^{\nu+1}\, dt \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs) K_\nu(st)\, s\, ds \qquad (\text{wo } x < \eta < x+\delta) \\ & = x^{\frac{1}{2}} \varphi_1(x+\delta) \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, s\, ds \int\limits_{\eta}^{x+\delta} K_\nu(st)\, t^{\nu+1}\, dt \\ & = x^{\frac{1}{2}} \varphi_1(x+\delta) \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs) \left[\eta^{\nu+1} K_{\nu+1}(\eta s) - (x+\delta)^{\nu+1} K_{\nu+1}\{(x+\delta)s\}\right] ds \end{aligned}\right\} \quad . \quad . \quad (44)$$

wegen (28).

---

22) Man beachte hierbei, dass das Integral

$$\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs) K_\nu(st)\, s\, ds$$

rein imaginär ist.

Das Integral mit $\varphi_2(t)$ kann auf dieselbe Weise behandelt werden.

Nun folgt aus (26), dass das erste Integral auf der rechten Seite von (43) gegen $\frac{1}{2}\pi i F(x+0)$ strebt für $\lambda \to \infty$. Der Beweis von (36) ist also geliefert, sobald gezeigt ist, dass die beiden auf der rechten Seite von (43) stehenden Integrale mit $\varphi_1(t)$ bezw. $\varphi_2(t)$ durch geeignete Wahl von δ beliebig klein gemacht werden können. Ich darf mich hierbei auf $\varphi_1(t)$ beschränken.

Nun ist $\varphi_1(x+\delta) < \varepsilon$ für δ hinreichend klein (siehe (42)). Mit Rücksicht auf (44) brauche ich also nur zu zeigen, dass das Integral

$$\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, K_{\nu+1}(\eta s)\, ds$$

beschränkt ist für alle positive Werte von λ und für $x < \eta \leqq x+\delta$, wobei η von λ abhängig sein darf. Nun sieht man leicht ein, wegen (13), (14) und (15), dass ich nur die Integrale

$$\int\limits_{\beta-\lambda i}^{\beta} \frac{e^{-xs-\eta s}\, ds}{s}, \quad \int\limits_{\beta}^{\beta+\lambda i} \frac{e^{-xs-\eta s}\, ds}{s}, \quad \int\limits_{\beta-\lambda i}^{\beta+\lambda i} \frac{e^{xs-\eta s}\, ds}{s}$$

zu betrachten brauche, und diese Integrale sind bezw. gleich

$$\int\limits_{(\beta-\lambda i)(x+\eta)}^{\beta(x+\eta)} \frac{e^{-\zeta}\, d\zeta}{\zeta}, \quad \int\limits_{\beta(x+\eta)}^{(\beta+\lambda i)(x+\eta)} \frac{e^{-\zeta}\, d\zeta}{\zeta}, \quad \int\limits_{(\beta-\lambda i)(\eta-x)}^{(\beta+\lambda i)(\eta-x)} \frac{e^{-\zeta}\, d\zeta}{\zeta},$$

so dass ich mich auf die Hilfssätze 6 und 5 berufen kann.

Hiermit ist (36) bewiesen.

Formel (37) folgt auf analoge Weise aus (41) und (27).

Beweis von (38). Wegen (16) hat man

$$\left.\begin{aligned} &\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,(xs)^{\frac{1}{2}}\, ds \int\limits_{0}^{x-\delta} K_\nu(st)\,(st)^{\frac{1}{2}}\, F(t)\, dt \\ &\qquad = x^{\frac{1}{2}} \int\limits_{0}^{x-\delta} F(t)\, t^{\frac{1}{2}}\, dt \int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, K_\nu(st)\, s\, ds = U_1 + U_2 - U_3 - U_4, \end{aligned}\right\} . \quad (45)$$

falls

$$U_1 = x^{\frac{3}{2}} (\beta+\lambda i)\, I_{\nu+1}\{(\beta+\lambda i)\, x\} \int\limits_{0}^{x-\delta} \frac{K_\nu\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2-t^2}$$

und

$$U_2 = x^{\frac{1}{2}}(\beta+\lambda i)\, I_\nu\{(\beta+\lambda i)\, x\} \int\limits_0^{x-\delta} \frac{K_{\nu+1}\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{3}{2}}\, dt}{x^2-t^2}$$

gesetzt wird und U_3 und U_4 die analogen Ausdrücke mit $\beta-\lambda i$ statt $\beta+\lambda i$ bezeichnen.

Ich betrachte nun für $\lambda \rightarrow \infty$ den Ausdruck

$$\left\{ \int\limits_0^{\lambda^{-1}} + \int\limits_{\lambda^{-1}}^{x-\delta} \right\} \frac{K_\nu\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2-t^2} \quad \ldots\ldots \tag{46}$$

Wegen

$$K_\nu(z) = \frac{\pi}{2 \sin \nu\pi}\{I_{-\nu}(z) - I_\nu(z)\}$$

gilt für das Integral von 0 bis λ^{-1}

$$\int\limits_0^{\lambda^{-1}} \frac{K_\nu\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2-t^2} = O\left\{\lambda^{-\nu}\int\limits_0^{\lambda^{-1}} |F(t)|\, t^{\frac{1}{2}-\nu}\, dt\right\}$$

$$+\, O\left\{\lambda^{\nu}\int\limits_0^{\lambda^{-1}} |F(t)|\, t^{\frac{1}{2}+\nu}\, dt\right\} = O\left\{\lambda^{-\frac{1}{2}}\int\limits_0^{\lambda^{-1}} |F(t)|\, dt\right\} = o(\lambda^{-\frac{1}{2}}).$$

Für $t \geqq \lambda^{-1}$ benutze ich (man vergl. (13))

$$K_\nu\{(\beta+\lambda i)\, t\} = \frac{\sqrt{\pi}\, e^{-\beta t - i\lambda t}}{\{2(\beta+\lambda i)\, t\}^{\frac{1}{2}}} + O\{(\lambda t)^{-\frac{3}{2}}\};$$

mit Hilfe dieser Formel findet man 23)

$$\int\limits_{\lambda^{-1}}^{x-\delta} \frac{K_\nu\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2-t^2} = \left(\frac{\pi}{2\beta+2\lambda i}\right)^{\frac{1}{2}} \left\{ \int\limits_0^{x-\delta} - \int\limits_0^{\lambda^{-1}} \right\} \frac{e^{-\beta t}\, F(t)\, e^{-i\lambda t}\, dt}{x^2-t^2}$$

$$+\, O\left\{\lambda^{-\frac{3}{2}}\int\limits_{\lambda^{-1}}^{\lambda^{-\frac{1}{2}}} |F(t)|\, t^{-1}\, dt\right\} + O\left\{\lambda^{-\frac{3}{2}}\int\limits_{\lambda^{-\frac{1}{2}}}^{x-\delta} |F(t)|\, t^{-1}\, dt\right\}$$

$$= o(\lambda^{-\frac{1}{2}}) + O\left\{\lambda^{-\frac{1}{2}}\int\limits_{\lambda^{-1}}^{\lambda^{-\frac{1}{2}}} |F(t)|\, dt\right\} + O\left\{\lambda^{-1}\int\limits_{\lambda^{-\frac{1}{2}}}^{x-\delta} |F(t)|\, dt\right\} = o(\lambda^{-\frac{1}{2}}).$$

---

23) Das Integral $\int\limits_0^{x-\delta} \frac{e^{-\beta t}\, F(t)\, e^{-i\lambda t}\, dt}{x^2-t^2}$ strebt gegen Null für $\lambda \rightarrow \infty$ wegen des RIEMANN-LEBESGUEschen Lemmas; man vergl. DOETSCH, [**4**], 50; TITCHMARSH, [**21**], 11.

Die beiden Bestandteile des Ausdrucks (46) sind daher $o(\lambda^{-\frac{1}{2}})$. Da wegen (15)

$$(\beta+\lambda i)\, I_{\nu+1}\{(\beta+\lambda i)\, x\} = O(\lambda^{\frac{1}{2}}) \quad . \; . \; . \; . \; . \; . \quad (47)$$

ist, strebt also U_1 gegen Null für $\lambda \to \infty$.

Genau so zeigt man, dass U_2, U_3 und U_4 dem Grenzwert Null zustreben, womit wegen (45) der Beweis von (38) geliefert ist.

Beweis von (39). Im Integral

$$\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,(xs)^{\frac{1}{2}}\, ds \int_{x+\delta}^{\infty} K_\nu(st)\,(st)^{\frac{1}{2}}\, F(t)\, dt$$

darf man die Integrationsfolge vertauschen; für $t > 0$ und $s = \beta + iy$ mit $-\lambda \leqq y \leqq \lambda$ gilt nämlich 24)

$$|\, K_\nu(st)\,(st)^{\frac{1}{2}}\,| \leqq \sqrt{\frac{\pi}{2}}\; e^{-\beta t},$$

so dass das Integral nach t wegen der zweiten Voraussetzung gleichmässig in s konvergiert. Nach der Vertauschung der Integrationsfolge bekommt man (man vergl. (16) und (45))

$$\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,(xs)^{\frac{1}{2}}\, ds \int_{x+\delta}^{\infty} K_\nu(st)\,(st)^{\frac{1}{2}}\, F(t)\, dt = V_1 + V_2 - V_3 - V_4, \qquad (48)$$

falls

$$V_1 = x^{\frac{3}{2}}(\beta+\lambda i)\, I_{\nu+1}\{(\beta+\lambda i)\, x\} \int_{x+\delta}^{\infty} \frac{K_\nu\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2-t^2}$$

und

$$V_2 = x^{\frac{1}{2}}(\beta+\lambda i)\, I_{\nu}\{(\beta+\lambda i)\, x\} \int_{x+\delta}^{\infty} \frac{K_{\nu+1}\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{3}{2}}\, dt}{x^2-t^2}$$

gesetzt wird und V_3 und V_4 die analogen Ausdrücke mit $\beta-\lambda i$ statt $\beta+\lambda i$ bezeichnen.

Nun hat man für $\lambda \to \infty$ mit Rücksicht auf (13)

$$\int_{x+\delta}^{\infty} \frac{K_\nu\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2-t^2} = \left(\frac{\pi}{2\beta+2\lambda i}\right)^{\frac{1}{2}} \int_{x+\delta}^{\infty} \frac{e^{-\beta t}\, F(t)\, e^{-i\lambda t}\, dt}{x^2-t^2}$$

$$+\, O\left\{\lambda^{-\frac{3}{2}} \int_{x+\delta}^{\infty} e^{-\beta t}\, |\, F(t)\,|\, t^{-1}\, dt\right\} = o(\lambda^{-\frac{1}{2}})$$

wegen des RIEMANN-LEBESGUEschen Lemmas.

---

24) Man vergl. Fussnote 3).

V_1 ist also $o(1)$ (siehe auch (47)). Ebenso zeigt man, dass auch V_2, V_3 und V_4 gegen Null streben für $\lambda \to \infty$, womit wegen (48) der Beweis von (39) erbracht ist.

§ 4. Beweis von Satz 2.

Wegen der Voraussetzung 2 ist

$$\left.\begin{aligned}
\sqrt{\frac{2}{\pi}} \int_0^b K_\nu(st)(st)^{\frac{1}{2}} F(t)\,dt &= \frac{1}{\pi i} \int_0^b K_\nu(st)(st)^{\frac{1}{2}}\,dt \lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(tz)(tz)^{\frac{1}{2}} f(z)\,dz \\
&= \frac{1}{\pi i} \lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} f(z)\,dz \int_0^b K_\nu(st)(st)^{\frac{1}{2}} I_\nu(tz)(tz)^{\frac{1}{2}}\,dt \\
= \frac{1}{\pi i} \lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} & \frac{\{z I_{\nu+1}(zb)(zb)^{\frac{1}{2}} K_\nu(sb)(sb)^{\frac{1}{2}} + s I_\nu(zb)(zb)^{\frac{1}{2}} K_{\nu+1}(sb)(sb)^{\frac{1}{2}}\} - s^{\frac{1}{2}-\nu} z^{\frac{1}{2}+\nu}}{z^2 - s^2} f(z)\,dz
\end{aligned}\right\} \quad (49)$$

wegen (16).

Nun hat man mit Rücksicht auf (13), (14) und (15) für $\Re(s) > \beta > 0$, $z = \beta + iy$ und $b \geqq 1$

$$|K_\nu(sb)(sb)^{\frac{1}{2}}| < D e^{-\beta b}, \quad |K_{\nu+1}(sb)(sb)^{\frac{1}{2}}| < D e^{-\beta b},$$
$$|I_\nu(zb)(zb)^{\frac{1}{2}}| < D e^{\beta b}, \quad |I_{\nu+1}(zb)(zb)^{\frac{1}{2}}| < D e^{\beta b},$$

wo D unabhängig von b ist; es gilt daher

$$|z I_{\nu+1}(zb)(zb)^{\frac{1}{2}} K_\nu(sb)(sb)^{\frac{1}{2}} + s I_\nu(zb)(zb)^{\frac{1}{2}} K_{\nu+1}(sb)(sb)^{\frac{1}{2}}| < D^2(|z| + |s|).$$

Der Integrand auf der rechten Seite von (49) ist also [25]) absolut genommen $< \dfrac{E\,|f(z)|}{|z|}$ für alle Werte von $z = \beta + iy$ mit $|y| \geqq L$; hierin sind E und L nicht von b abhängig.

Das auf der rechten Seite von (49) stehende Integral ist daher mit Rücksicht auf der dritten Voraussetzung für alle Werte von $b \geqq 1$ gleichmässig in b konvergent, so dass man b unter dem Integralzeichen gegen ∞ streben lassen darf. Das Resultat wird wegen (13), (14) und (15), da $\Re(s) > \beta$ ist,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty K_\nu(st)(st)^{\frac{1}{2}} F(t)\,dt = \frac{s^{\frac{1}{2}-\nu}}{\pi i} \lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} \frac{z^{\frac{1}{2}+\nu} f(z)\,dz}{s^2 - z^2} \quad . \quad . \quad . \quad (50)$$

Ich betrachte nun das Integral

$$\frac{s^{\frac{1}{2}-\nu}}{\pi i} \int \frac{z^{\frac{1}{2}+\nu} f(z)\,dz}{z^2 - s^2},$$

---

[25]) Man beachte auch, dass $\Re(\nu) \leqq \frac{1}{2}$ ist.

im positiven Sinne erstreckt über den Rand des Rechtecks mit den Ecken $\beta + \lambda i$, $\beta - \lambda i$, $\lambda - \lambda i$, $\lambda + \lambda i$. Nach dem Residuensatze besitzt dieses Integral für hinreichend grosses λ den Wert $f(s)$.

Nun kann man beweisen [26]) unter Benutzung der vierten und fünften Voraussetzung, dass die auf die Seiten $(\lambda - \lambda i, \lambda + \lambda i)$, $(\lambda + \lambda i, \beta + \lambda i)$ und $(\beta - \lambda i, \lambda - \lambda i)$ bezüglichen Integrale gegen Null streben für $\lambda \to \infty$.

Die rechte Seite von (50) hat daher den Wert $f(s)$, womit der Beweis von Satz 2 geliefert ist.

§ 5. Schlussbemerkung.

Verschiedene Spezialfälle der Relationen (8) und (9) kommen vor in meinen früheren Arbeiten. Ich werde fünf Beispiele anführen:

Erstes Beispiel [27]):

$$H_\nu^{(1)}(z)\, H_\nu^{(2)}(z) = \frac{8z}{\pi} \int_0^\infty K_{2m}(2zu)\, P_{\nu-\frac{1}{2}}^{m}(\sqrt{1+u^2})\, P_{\nu-\frac{1}{2}}^{-m}(\sqrt{1+u^2})\, u\, du$$

$$[|\arg z| < \tfrac{1}{2}\pi;\ |\Re(m)| < 1],$$

$$P_{\nu-\frac{1}{2}}^{m}(\sqrt{1+u^2})\, P_{\nu-\frac{1}{2}}^{-m}(\sqrt{1+u^2}) = \frac{1}{2i} \int_{\beta-\infty i}^{\beta+\infty i} I_{2m}(2uz)\, H_\nu^{(1)}(z) H_\nu^{(2)}(z)\, dz \qquad [u > 0].$$

Zweites Beispiel [28]):

$$\left.\begin{aligned} P_{\nu-\frac{1}{2}}^{m}(\sqrt{1+z^2})\, P_{\nu-\frac{1}{2}}^{-m}(\sqrt{1+z^2}) = \frac{4\cos\nu\pi}{\pi} \int_0^{\infty e^{-i\arg z}} K_{2\nu}(2v)\, J_m(zv)\, J_{-m}(zv)\, dv \\ [|\arg z| < \tfrac{1}{2}\pi;\ |\Re(\nu)| < \tfrac{1}{2}], \end{aligned}\right\} \quad (51)$$

$$\left.\begin{aligned} J_m(\zeta) J_{-m}(\zeta) = \frac{1}{i\cos\nu\pi} \int_{\beta-\infty i}^{\beta+\infty i} I_{2\nu}(2v)\, P_{\nu-\frac{1}{2}}^{m}\left(\sqrt{1+\frac{\zeta^2}{v^2}}\right) P_{\nu-\frac{1}{2}}^{-m}\left(\sqrt{1+\frac{\zeta^2}{v^2}}\right) dv \\ [\zeta > 0;\ \nu \neq \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \ldots]. \end{aligned}\right\} \quad (52)$$

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[26]) Man vergl. einen analogen Beweis in dem entsprechenden DOETSCH-CHURCHILLschen Satz; DOETSCH, [**4**], 127—128; CHURCHILL, [**3**], 570—571.

[27]) [**14**], Formel (76) und [**11**], Formel (24).

[28]) [**13**], Formel (12) und [**10**], Formel (19).

(51) und (52) gehen über in Formeln der Gestalt (8) und (9), wenn man $z = \frac{1}{s}$ und $v = \frac{1}{2} st$ substituiert in (51) und überdies ζ durch $\frac{1}{2}t$ und v durch $\frac{1}{2}ts$ ersetzt in (52).

Drittes Beispiel[29]:

$$K_{2m}(z\,e^{\frac{1}{4}\pi i})\,K_{2m}(z\,e^{-\frac{1}{4}\pi i}) = \frac{1}{z^2}\int\limits_0^{\infty\, e^{2i\arg z}} K_{2k}(v)\,W_{k,m}\left(\frac{z^2}{2v}\right)W_{-k,m}\left(\frac{z^2}{2v}\right)v\,dv, \tag{53}$$

$$W_{k,m}(\tfrac{1}{2}\zeta)\,W_{-k,m}(\tfrac{1}{2}\zeta) = \frac{\zeta}{\pi i}\int\limits_{\beta-\infty i}^{\beta+\infty i} I_{2k}(v)\,K_{2m}(\zeta^{\frac{1}{2}}v^{\frac{1}{2}}e^{\frac{1}{4}\pi i})\,K_{2m}(\zeta^{\frac{1}{2}}v^{\frac{1}{2}}e^{-\frac{1}{4}\pi i})\,dv. \tag{54}$$

Viertes Beispiel[30]:

$$G^{4,0}_{0,4}(\tfrac{1}{16}s^2 \mid a, q, \tfrac{1}{2}\nu, -\tfrac{1}{2}\nu) = 2^{3-a-q}\,s^{a+q}\int\limits_0^{\infty} K_{a-q}(st)\,K_\nu\left(\frac{1}{t}\right)t^{a+q-1}\,dt, \tag{55}$$

$$K_\nu\left(\frac{1}{t}\right) = \frac{2^{a+q-3}\,t^{2-a-q}}{\pi i}\int\limits_{\beta-\infty i}^{\beta+\infty i} I_{a-q}(ts)\;G^{4,0}_{0,4}(\tfrac{1}{16}s^2 \mid a, q, \tfrac{1}{2}\nu, -\tfrac{1}{2}\nu)\,s^{1-a-q}\,ds. \tag{56}$$

Das fünfte Beispiel[31] ist ein Spezialfall des vierten $(a = \frac{1}{2}\nu + \frac{1}{2},\ q = -\frac{1}{2}\nu + \frac{1}{2})$

$$K_{2\nu}(2s^{\frac{1}{2}}) = \frac{s}{\pi}\int\limits_0^{\infty} K_\nu(st)\,K_\nu\left(\frac{1}{t}\right)dt, \tag{57}$$

$$K_\nu\left(\frac{1}{t}\right) = \frac{t}{i}\int\limits_{\beta-\infty i}^{\beta+\infty i} I_\nu(ts)\,K_{2\nu}(2s^{\frac{1}{2}})\,ds. \tag{58}$$

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[29] [**12**], 26 und [**9**], 875, Formel (12). In (53) ist $|\arg z| < \frac{1}{4}\pi$; (54) gilt für $|\arg\zeta| < \frac{1}{2}\pi$.

[30] [**12**], 24, Satz 8 und [**15**], Formel (5). Für die Definition der Funktion $G^{4,0}_{0,4}$ vergl. man [**12**], Formel (6). In (55) und (57) ist $|\arg s| < \frac{1}{2}\pi$; in (56) und (58) wird $t > 0$ vorausgesetzt.

[31] Man vergl. HARDY, [**7**], Formel (4. 1) und MEIJER, [**15**], Formel (35).

Mathematics. — *On* LAMBERT'S *proof for the irrationality of* π. By J. POPKEN. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of May 25, 1940.)

In their beautiful book "An introduction to the theory of numbers" HARDY and WRIGHT state, there is no simple proof for the irrationality of π [1]). Yet, if LAMBERT'S classical proof [2]) is freed from the continued fraction algorithm, it takes a surprisingly easy form [3]). We only require the following lemma:

Lemma: Let $x \neq 0$. For $h = 0, 1, 2, \ldots$ there are polynominals $p_h(x^{-1})$ and $q_h(x^{-1})$ in x^{-1} with integral coefficients and of degree at most $2h$, so that

$$\left.\begin{aligned} & p_h(x^{-1}) \sin x + q_h(x^{-1}) \cos x = \\ & = (-2)^h \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)\ldots(n+h)}{(2n+2h+1)!} x^{2n+1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

Proof: a. $h = 0$. Take $p_0(x^{-1}) \equiv 1$, $q_0(x^{-1}) \equiv 0$, then (1) is identical with

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots .$$

b. Supposing (1) to be true for h, then we prove, that (1) also holds for $h+1$ instead of h.

In fact, if (1) is true, then

$$x^{-1} p_h(x^{-1}) \sin x + x^{-1} q_h(x^{-1}) \cos x =$$

$$= (-2)^h \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)\ldots(n+h)}{(2n+2h+1)!} x^{2n}$$

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1) p. 39 and p. 47.

2) J. H. LAMBERT, Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmétiques. Histoire Acad. roy. des sciences et belles lettres. Berlin. Année 1761 (1768), p. 265—322.

3) Nearly the same idea has been applied by HERMITE: Sur quelques approximations algébriques, Oeuvres III, p. 146—149, but HERMITE used an integral.

Differentiation gives

$$\left.\begin{aligned}&\left[\frac{d}{dx}\{x^{-1}\,p_h(x^{-1})\}-x^{-1}\,q_h(x^{-1})\right]\sin x+\\&\qquad+\left[x^{-1}\,p_h(x^{-1})+\frac{d}{dx}\{x^{-1}\,q_h(x^{-1})\}\right]\cos x\\&=(-2)^h\sum_{n=1}^{\infty}(-1)^n\frac{2n(n+1)(n+2)\ldots(n+h)}{(2n+2h+1)!}\,x^{2n-1}=\\&=(-2)^{h+1}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{(\nu+1)(\nu+2)\ldots(\nu+h+1)}{(2\nu+2h+3)!}\,x^{2\nu+1}\end{aligned}\right\}\quad .\ .\quad (2)$$

Putting

$$\left.\begin{aligned}\frac{d}{dx}\{x^{-1}\,p_h(x^{-1})\}-x^{-1}\,q_h(x^{-1})&=p_{h+1}(x^{-1}),\\x^{-1}\,p_h(x^{-1})+\frac{d}{dx}\{x^{-1}\,q_h(x^{-1})\}&=q_{h+1}(x^{-1}),\end{aligned}\right\}\quad .\ .\ .\ .\quad (3)$$

it is clear, that $p_{h+1}(x^{-1})$ and $q_{h+1}(x^{-1})$ are polynomials in x^{-1} with integral coefficients and of degree at most $2h+2$, since $p_h(x^{-1})$ and $q_h(x^{-1})$ are polynomials, whose coefficients are integers and are of degree $2h$ at most. By (2) and (3)

$$p_{h+1}(x^{-1})\sin x+q_{h+1}(x^{-1})\cos x=$$

$$=(-2)^{h+1}\sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)\ldots(n+h+1)}{(2n+2h+3)!}\,x^{2n+1}.$$

Hence part *b.* of our proof is evident.

Proof of LAMBERT's *theorem:* If π is rational, then $\frac{\pi}{4}=\frac{a}{b}$, where a and b are positive integers. Applying the preceding lemma with $x=\frac{\pi}{4}=\frac{a}{b}$, we derive for any integer $h\geqq 0$

$$\left\{p_h\left(\frac{b}{a}\right)+q_h\left(\frac{b}{a}\right)\right\}\tfrac{1}{2}\sqrt{2}=$$

$$=(-2)^h\left(\frac{h!}{(2h+1)!}\,\frac{\pi}{4}-\frac{2\,.\,3\ldots(h+1)}{(2h+3)!}\left(\frac{\pi}{4}\right)^3+\ldots\right).$$

In the alternating series between brackets the terms decrease in absolute value and tend to zero. Hence

$$2^h\left(\frac{h!}{(2h+1)!}\,\frac{\pi}{4}-\frac{2.3\ldots(h+1)}{(2h+3)!}\left(\frac{\pi}{4}\right)^3\right)<$$

$$<\tfrac{1}{2}\sqrt{2}\left|p_h\left(\frac{b}{a}\right)+q_h\left(\frac{b}{a}\right)\right|<2^h\frac{h!}{(2h+1)!}\,\frac{\pi}{4}<\frac{2^h}{h!},$$

$$0<\left|a^{2h}\,p_h\left(\frac{b}{a}\right)+a^{2h}\,q_h\left(\frac{b}{a}\right)\right|<\sqrt{2}\,\frac{2^h\,|\,a\,|^{2h}}{h!}.\;\;.\;\;.\quad(4)$$

Since $p_h(x^{-1})$ is a polynomial in x^{-1} with integers as coefficients and of degree say g the number $p_h\left(\frac{b}{a}\right)$ can be written as a fraction with denominator a^g. But $g \leqq 2h$, therefore $a^{2h}\,p_h\left(\frac{b}{a}\right)$ is an integer and the same reasoning shows, that $a^{2h}\,q_h\left(\frac{b}{a}\right)$ also is an integer.

Hence for integral $h \geqq 0$

$$\left|a^{2h}\,p_h\left(\frac{b}{a}\right)+a^{2h}\,q_h\left(\frac{b}{a}\right)\right|$$

is a positive integer. But now (4) leads to a contradiction, since

$$\lim_{h\to\infty}\sqrt{2}\,\frac{2^h\,|\,a\,|^{2h}}{h!}=0.$$

Botany. — *Nitrate Assimilation of Aspergillus niger* VAN TIEGHEM. By S. DE BOER. (Communicated by Prof. J. C. SCHOUTE.)

(Communicated at the meeting of May 25, 1940.)

Introduction.

The object of this investigation was the study of nitrate metabolism of Aspergillus niger, for the purpose of ascertaining:

a. whether nitrate, after being taken up, is accumulated by the fungus or is directly used in metabolism, and

b. the influence of the pH on these processes.

Further, the influence of glucose was investigated, as it might be expected that glucose will have a similar effect in the case of Aspergillus as of Avena roots where the nitrate uptake und nitrate reduction is increased by the addition of glucose, as POSTMA (7) has proved.

For this investigation the nitrate had to be quantitatively determined in the fungus as well as in the solution. According to the literature the determination of nitrate by the diphenylamine and phenoldisulphonic acid methods has not yielded satisfactory results. ITZEROTT (5) and BÜNNING (4) used the brucine method in determining the nitrate in a culture solution, whereas in ascertaining the mycelium only qualitative observations were made by the diphenylaminesulphuric acid method. Respecting the brucine method, according to ALTEN and WEILAND (1), account must be taken of great variations (as much as 25 %), and although ITZEROTT and BÜNNING state that they have obtained more favourable results, it will be advisable to employ a better method. The xylenol method, first applied by BLOM and TRESCHOW (3) to soil and plants, is a more exact nitrate determination which has been employed in different researches of recent years. This method, which I followed in determining the nitrate in the culture solution and, in a modified form, also for fungi, yielded satisfactory results.

In physiological experiments concerning nitrate assimilation by fungi, the usual procedure is as follows. The fungi are reared on a medium containing the substance to be examined, in this case nitrate. After some days the quantity still remaining of the nitrate first added to the medium is then ascertained. The difference is caused by the nitrate accumulated in the fungus and the nitrate taken up in metabolism partly for building up new protoplasm. In these circumstances it is not possible to separate these two processes. For this reason I employed fungus mats which, moreover, were starved on water. This method, which has been employed by VAN WAESBERGHE (8) among others, and which suits the fungus, as his experiments show, has the advantage that the growth-phase is separated from the investigation-phase. Moreover, by starving, the reserve substances which might affect the nitrate metabolism, are used up.

§ 1. *Experimental Methods.*

a. *Culture method.*

In a 100 ml Erlemeyer-flask containing 37 ml nitrate-free Raulin-solution, a spore suspension of Aspergillus niger is sown. The fungus is reared for 2 days, after which the mats are starved on water. After one day the water is replaced by a known nitrate solutioa in which, after some time, the nitrate which has not been taken up by the fungus, is determined.

b. *Nitrate determination.*

The nitrate determination in the solution was carried out, with slight modifications, after the directions given by ALTEN et al. (2) and WERR (9):

To 5 ml solution (which must not contain more than 4 mg NO_3) 25 ml 75 vol. % nitrate-free H_2SO_4 is added and, after cooling to about 25°, 3 drops 2 : 4-xylen-1-ol. The flask is closed and well shaken, after which the solution must stand half an hour in the dark, in this time the xylenol is nitrated to nitroxylenol. The solution is then cooled in ice-water and diluted with 100 ml water. The diluted nitroxylenol solution may be kept in the dark till it is distilled with steam. Some nitroxylenol always crystallizes in the condensor; this is removed by shutting off the water of the condensor for some time in order that the crystals may be carried away with the steam. About 70 ml of the distillate is collected in 5 ml 1 % NaOH, in which it forms a yellowish-brown product. Water is added to 100 ml and the extinction of this liquid is determined in a Zeiss Stufenphotometer with S 47 filter.

A standard-curve is made by determining from different nitrate concentrations the respective extinction-values. Between 0.2 and 4.0 mg NO_3 a practically straight line can be obtained. In this way nitrate quantities of 0.2 to 4.0 mg per 5 ml can be determined with 1—2 % of accuracy.

For a nitrate determination in the fungus, it is necessary, according to MCVEY's (6) experiments with meat, to dissolve the nitrate first in water and to remove the protein. This is done in the following way: 2 fungus mats were ground with quartz-sand in a mortar, transferred to a beaker with 20 ml of water, made slightly alkaline with NaOH (indicator litmus) and then heated on a water-bath for an hour. Then this suspension is brought to a volume of 25 ml, and centrifuged. 15 ml of the liquid is acidified with 10 % H_2SO_4 to pH 3.6 (yellow to bromo-cresol green), after which 1 ml 20 % phosphotungstic acid solution is added to precipitate the protein [1]). In a measuring flask this mass was filled up to 25 ml, centrifuged and 5 ml of the liquid was used for the determination of the nitrate. To determine whether the nitrate-quantum has not increased or diminished by these manipulations, a known quantity of nitrate was added to nitrate-free fungus at different stages of the process, and at the end the quantity of nitrate is determined. These values proved to differ from the original by not more than 2 %.

It further appeared that when equal quantities of nitrate are added to fungi all cultivated and starved in the same way, about equal quantities are recovered some days later in the several series (see Table I).

TABLE I. Aspergillus niger, reared for 2 days on a nitrate-free RAULIN-solution, starved for 1 day, thereafter 1 or 2 days on 40 ml phosphate buffer pH 6 with KNO_3. Figures are mg per 1 fungus mat.

Added mg NO_3	25.2	25.2	25.2	25.2
Found in solution after 1 day	23.0	23.8	23.1	23.1
„ „ fungus „ „ „	1.4	1.4	1.4	1.4
„ „ solution „ 2 days	22.8	22.5	22.7	22.7
„ „ fungus „ „ „	0.8	0.6	0.5	0.4

The differences in these four series of 2 fungus mats are slight. This method is therefore well suited for determining nitrate in fungi and their solutions.

§ 2. *Influence of the pH upon the nitrate assimilation.*

a. *With starved fungus mats.*

The fungi, after being starved, are transferred to sterile phosphate buffers, to which nitrate has been added in the form of KNO_3. The pH changes but little and is restored to the original value again by the addition of phosphoric acid or NaOH. The results of the various experiments have been collected in Table II.

The table shows that nitrate has been taken up by the fungus from the solutions. In the fungus there is always a very small quantity of nitrate and thus no question of nitrate accumulation. No distinct connection can be observed between the pH and the quantum NO_3 taken up.

ITZEROTT (5) and BÜNNING (4) concluded from their experiments that the nitrate assimilation is dependent upon the pH, and observed a nitrate accumulation in the cells if the pH of the solution is lower than 3. But, as

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[1] In this case the protein was not precipitated with trichloro-acetic acid, as indicated by GORTER (10), because Cl has a disturbing effect in the xylenol method for nitrate determination.

TABLE II. Aspergillus niger, reared for 2 days on nitrate-free RAULIN-solution, 1 day starved, thereafter 1 or 2 days on 40 ml phosphate buffer of different pH, with nitrate, the original quantity of which is given in mg. The quantities recovered given in percentages of the quantum added. Figures per 1 fungus mat.

Exp. Nr	Duration of experiment after starvation	Original nitrate quantity mg	Nitrate recovered in % of original quantity	pH 2	pH 4	pH 6	pH 8
26	1 day	45.2	in solution	98.0	91.8	96.5	96.0
			in fungus	0.4	1.8	1.8	1.3
27	1 day	48.3	in solution	98.1	95.8	93.2	
			in fungus	1.8	2.9	5.3	
	2 days		in solution	93.8	97.5	93.8	
			in fungus	2.5	0.2	0.4	
28	2 days	58.5	in solution	84.5	93.2	92.2	
			in fungus	0.8	0.8	0.6	
31	2 days	55.2	in solution	80.0	82.2	86.6	86.1

was remarked above, these authors made no quantitative determinations in the fungus mats, but merely determined nitrate in the buffer solution in experiments with non-starved fungi.

b. *With growing fungi.*

In the case of non-starved, or still growing, fungus mats, nitrate is much more used in metabolism than in the case of starved fungi. Here the pH may have some influence; some experiments were therefore carried out respecting the nitrate assimilation with growing fungi. For this purpose the fungi were reared at least 3 days in Raulin-solutions of different pH to which 5 g KNO_3 per l, had been added. During the growth it was observed that the fungi at high pH formed but very thin mats. After a few days the nitrate quantum in the solutions and fungi, as also the dry-weights, were determined. (See Table III.)

TABLE III. (Exp. nr. 30). Aspergillus niger, reared on 37 ml nitrate containing RAULIN-solution of different pH. Original quantity of nitrate 108,9 mg. Recovered quantities of nitrate in percentages of the original. Figures per 1 fungus mat.

Duration of experiment	Recovered nitrate in %	pH 2	pH 4	pH 6	pH 8
4 days	in solution	54.0	12.0	26.6	75.8
	in fungus	0.0	0.0	0.0	0.0
5 days	in solution	0.0	0.0	0.0	10.5
	in fungus	0.1	0.0	0.0	0.0
4 days	dry weight in mg	184	267	273	117
5 days	dry weight in mg	387	517	457	344

It follows from this experiment that nitrate is quickest taken up by the fungi at pH 4. Nowhere has nitrate been found in the fungus, it is therefore not accumulated. The nitrate when taken up is at once metabolised, and this also takes place most rapidly at pH 4. Likewise the dry-weight is greatest at pH 4. We may thus state that the growth of the fungi is dependent upon the pH and that parallel to this is the nitrate assimilation (the uptake, accumulation and further transformation of nitrate together).

§ 3. *Influence of glucose.*

In contrast with § 2 where either a certain quantum of nitrate was used with starved fungi, or where non-starved fungi were reared on solutions containing nitrate and glucose, in the following experiments both nitrate and glucose were added to the starved fungus mats. By this method it could be ascertained whether carbohydrates are necessary as a source of energy in nitrate assimilation. To bring out the effect of glucose in the two series of experiments given below, different quantities of glucose were used. (See Table IV.)

TABLE IV. Aspergillus niger, reared for 2 days on nitrate-free RAULIN-solution, 1 day starved; thereafter 1 or 2 days on 40 ml phosphate buffer of different pH with nitrate and glucose. The nitrate quantum originally added in mg. The quantities recovered in percentages of the quantum added. Figures per 1 fungus mat.

Exp. Nr	Duration of experiment after starvation	Glucose added mg	Original quantity of nitrate mg	Nitrate recovered in % of original quantity	pH 2	pH 4	pH 6	pH 8
28	2 days	300	58.5	in solution	47.5	54.9	55.0	
				in fungus	0.6	0.2	0.2	
31	1 day	30	55.2	in solution	83.0	85.7	88.4	86.6
	2 days			in solution	71.5	82.4	82.2	79.1

Compare with these the corresponding values without glucose from Table II, experiment Nrs. 28 and 31.

It appears from these experiments that the addition of glucose promotes the assimilation of nitrate, for in this case more nitrate is taken up and metabolised. There is no accumulation here either, as can be seen from the first experiment. At pH 2 the influence of glucose in these cases is slightly greater.

The dry-weights of the fungi, with or without glucose show at different pH practically no differences. In contrast to the non-starved (§ 2 *b*), these fungi have been reared on a solution of the same pH, and the substances added after starvation have apparently had little influence on their dry-weight.

The influence of glucose is particularly noticeable when a larger quantity is added, for it appears that more glucose has a greater influence in promoting the nitrate metabolism. A similar influence of glucose was seen in the experiments of POSTMA (7) with Avena roots. A determination of the respiration in the case of Aspergillus will have to decide whether here, where the addition of glucose promotes the nitrate assimilation, there also will be an increased respiration.

In conclusion I wish to express my thanks to Dr. A. W. H. VAN HERK, and Mr. A. GORTER for their interest in this research.

Summary.

1. The xylenol method for determining the nitrate in fungi and their culture solutions yields satisfactory results.

2. Nitrate is not accumulated in Aspergillus niger, but is metabolised after uptake.

3. In the case of starved fungus mats, the pH is not seen to have any effect upon the nitrate assimilation.

4. In growing fungi the nitrate assimilation is greatest at pH 4 in connection with their growth.

5. Nitrate assimilation is increased by addition of glucose.

LITERATURE.

1. F. ALTEN u. H. WEILAND, Z. Pflanzenern., Düngung u. Bodenk. A. **32**, 337 (1933).
2. F. ALTEN, B. WANDROWSKY u. E. HILLE, Bodenk. u. Pflanzenern. **1**, 46 (1936).
3. J. BLOM u. C. TRESCHOW, Z. f. Pflanzenern., Düngung u. Bodenk. A. **13**, 159 (1929).
4. E. BÜNNING, Flora **31**, 87 (1936).
5. D. ITZEROTT, Flora **31**, 60 (1936).
6. W. C. MCVEY, J. Assoc. off. agric. Chemists **18**, 143 and 459 (1935).
7. W. P. POSTMA, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 181 (1939).
8. H. P. J. M. VAN WAESBERGHE, Rec. trav. bot. néerl. **34**, 650 (1937).
9. F. WERR, Z. Wirtschaftsgr. Zuckerind. Techn. T. **87**, 119 (1937).
10. A. GORTER, Thesis, Amsterdam, 1940 (in the press).

Amsterdam, Laboratory of plant physiology
of the University.

Botany. — *Amino acid Breakdown by Aspergillus niger.* By A. GORTER. (Communicated by Prof. J. C. SCHOUTE.)

(Communicated at the meeting of May 25, 1940.)

Introduction.

Little is known respecting the way in which amino acids are broken down by fungi. VAN WAESBERGHE (11) demonstrated that l(—)aspartic acid can be deaminated by Aspergillus niger, in which process oxygen is taken up, and MOOI (8) found that dl-alanine can also be deaminated by this fungus.

On the other hand, of recent years a number of enzymes have been found, which play a part in the amino acid breakdown by animal tissues and bacteria. The most important of these enzymes are given in Table I.

TABLE I.

Name	Reaction	Experimental conditions	Occurrence	Author
Aspartase I and II	l (—)aspartic acid $\rightleftarrows$ NH_3 + fumaric acid	Opt. pH: 6.0—7.5 Temp.: 37° anaerobic	Various bacteria	VIRTANEN and TARNANEN (10) GALE (4)
d-amino-oxidase	d-amino acid + $\frac{1}{2} O_2$ $\rightarrow NH_3$ + keto acid	Opt. pH: 8.5—9.0 Temp.: 37° aerobic	Kidney and liver extracts	KREBS (7) WARBURG and CHRISTIAN (12)
l-amino-oxidase	l-amino acid + $\frac{1}{2} O_2$ $\rightarrow NH_3$ + keto acid	Opt. pH: 7.4 Temp.: 37° aerobic	Slices of liver and kidney; bacteria	KREBS (7)
glutamic acid dehydrogenase	l(+)glutamic acid + $H_2O - 2H \rightleftarrows NH_3$ + keto-glutaric acid	Opt. pH: 7.4 Temp.: 30° with as H_2 acceptor: methylene blue	Extr. of animal and plant tissues; bacteria; yeast	VON EULER et al. (3)
aminopherase	l-amino acid′ + keto acid′ $\rightleftarrows$ keto acid″ + l-amino acid″	Opt. pH: 7.4 Temp.: 37° aerobic and anaerobic	extr. of muscle tissue; bacteria; plant tissue	BRAUNSTEIN and KRITZMANN (1,2)

I have now investigated whether one or more of these enzymes might be of importance in the breakdown of amino acids by Aspergillus niger. A more detailed survey of the results obtained will be published shortly (5).

§ 1. *Methods.*

All the experiments were carried out with *starved fungus mats*. These are mats which are reared on a culture-solution (RAULIN's solution was used in every case), and then placed on water for 10—20 hours. By this method not only the synthetic reactions were suppressed as far as possible and the breakdown reactions were promoted, but, moreover, starved mats can decompose amino acids more easily than when not starved, as VAN WAESBERGHE (l.c.) has demonstrated.

After the starvation, the water used for this was replaced by a phosphate buffer to which was added the amino acid to be examined, that had been brought to the same pH beforehand.

The culture of the mats, the starvation and the experiment itself was carried out in flasks in which the quantity of oxygen absorbed, and of carbon dioxide produced, could be determined at the same time, during the experiment. For this purpose a certain quantity of alkali was put into the tube A of the flask employed (Fig. 1), in which the carbon dioxide was absorbed which could be determined after the experiment. During the experiment the oxygen uptake caused a decrease in volume, which was measured in a gas-burette connected with the flask.

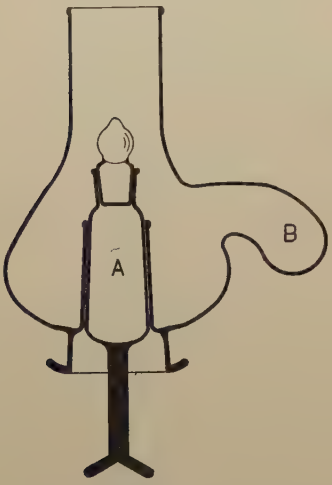

Fig. 1.

The substrate solution could be added when desired from the side-tube B, as is also usual with the Warburg method. The volume of the flasks was about 100 ml; during the experiment they were shaken in a thermostate at 25°, after being filled with O_2. A number of fungus mats was always investigated at the same time in order to eliminate the physiological variability as far as possible.

The starved mats show on a phosphate buffer also a respiration of their own; this must be subtracted from the respiration on phosphate buffer plus substrate in order to obtain the actual O_2 uptake and CO_2 production brought about by the substrate.

§ 2. *Deamination of aspartic acid.*

In the first place I investigated the influence of the pH on the rate of deamination. For this purpose the starved mats were brought on phosphate buffers of different pH, to which l(—)aspartic acid had been added to a final concentration of 0.025 mol. As a measure of the deamination rate, the quantity of NH_3 produced during the first 4 hours was determined. It appears now, from Table 2, that the results of the various experiments differ considerably, but that there is a distinct optimum at pH 2—4. For practical reasons the following experiments were always carried out at pH 4.5—5.0.

TABLE 2. Aspergillus mats reared for 2 days and thereafter starved for 12 hours. For pH 1.0 and 1.5: KCl/HCl buffer; pH 2.0 and 4.0: KH_2PO_4/H_3PO_4 buffer; pH 6.0 and 8.0: Na_2HPO_4/KH_2PO_4 buffer, always 0.1 mol. + 0.025 mol. l(—)aspartic acid. With experiment 63 and 65 10 ml buffer, and shaken in O_2 atmosphere; with experiment 66 20 ml buffer, not shaken, in air atmosphere. Temperature 25°. The figures show mg NH_3-N per mat, formed after 4 hours.

pH / Exper. nr.	1.0	1.5	2.0	4.0	6.0	8.0
63	—	—	0.83	0.59	0.43	0.0
65	0.0	—	1.92	0.91	—	—
66	0.0	0.42	1.40	1.28	0.83	0.21

It is remarkable that the deamination proceeds so rapidly at so low a pH of the buffer solution, whereas the deamination enzymes described, all have a much higher pH optimum (Cf. Table 1). It is true that the pH *in* the cells is different, and probably higher, than in the surrounding buffersolution, while permeability and further transformations of the deamination product also play a part. Yet it is improbable that the pH, *in* the cell, will be near 7.5, for at this pH of the surrounding solution there is hardly any more deamination at all. This is an argument against the activity of the enzymes above-mentioned.

When the deamination and oxygen uptake were measured during the experiment it appeared that at first amino acid disappeared without much O_2 being taken up. Not until later is there a relative increase in the O_2 uptake (Table 3); and after complete deamination approx. 3—4 atoms of oxygen were taken up; VAN WAESBERGHE (l.c.) had already found a value of about 3 atoms. If the breakdown process be continued longer

TABLE 3. Aspergillus mats, 17 hours starved, thereafter on 10 ml phosphate buffer pH 5.0 with 0.024 mol. l(—)aspartic acid. After various lengths of time amino acid determined in solution. O_2 atmosphere. Temperature 25°. Figures mg per fungus mat.

Time in hours	Aspartic acid present mg	Aspartic acid disappeared mg	O_2 uptake ml 0°/76 cm	O_2 uptake in at. p. mol. asp. acid disappeared
0	31.7	—	—	—
2	23.7	8.0	0.53	0.8
4	15.9	15.8	2.24	1.7
6	12.9	18.8	4.37	2.8
10	5.8	25.9	6.67	3.0
24	2.3	29.4	9.82	4.0

the oxygen uptake proceeds but slowly, so that when 3—4 at. of oxygen per mol. aspartic acid has been taken up, the oxidation may be regarded as ended, the CO_2 formed being then approx. 2—2.6 mol. There is thus never complete oxidation. Moreover, the experiment shows that at first deamination proceeds more rapidly that the subsequent oxidation of the deamination product.

§ 3. *Deamination of other amino acids.*

With respect to a number of other amino acids I now investigated whether they were deaminated under the same experimental conditions as aspartic acid, i.e. at pH 4.5—5.0 and in O_2 atmosphere. This proved to be the case with glycocoll, d(—)- and l(+)alanine; dl-aspartic acid; l(+)glutamic acid, dl-leucine and dl-isoleucine; other amino acids were not investigated. None of the amino acids mentioned was oxidised completely, as is shown in Table 4. The deamination itself was complete, except in the case of dl-leucine and dl-isoleucine.

TABLE 4. Starved Aspergillus mats on phosphate buffer pH 4.5—5.0 + 0.025 mol. amino acid. O_2 uptake and CO_2 output calculated per mol. transformed amino acid. Averages of a number of experiments.

Substrate	O_2 uptake in at. p. mol. substrate	CO_2 production in mol. p. mol. substrate
l (—) aspartic acid	3.4	2.2
dl-aspartic acid	3.0	—
l (+) alanine	3.4	—
d (—) alanine	3.0	1.5
dl-alanine	3.1	—
glycocoll	1.5	0.8
l (+) glutamic acid	5.3	2.5
dl-leucine	5.1	—
dl-isoleucine	4.3	—

With the amino acids examined the deamination proceeded as a rule in the same way as with aspartic acid, i.e., at first the amino acid disappeared, while but little oxygen is taken up; later the oxygen uptake increased proportionately. The value found for the oxygen uptake and the carbon dioxide output are no indication respecting the way in which the amino acids are broken down or respecting the enzymes which are active thereby.

This last was now further studied by examining what was the first product of the deamination.

§ 4. *The enzymes which play a part in the deamination.*

When *aspartase* plays a part in the aspartic acid breakdown the deamination must also take place anaerobically, and fumaric acid must be demonstrable. This was not the case, however; in nitrogen atmosphere there is no deamination at all. It is possible that the permeability will be much reduced in nitrogen atmosphere and the aspartic acid assimilation therefore retarded, since Aspergillus is a highly aerobic organism. In an aerobic experiment any fumaric acid formed may disappear, however, by oxidation, and therefore Na-arsenite was added to inhibit the oxidation. Fumaric acid was determined according to the method of HAHN and HAARMANN (6) by precipitation as mercuric salt, but in not a single experiment could fumaric acid be demonstrated as a product of the aspartic acid breakdown.

From this it may be concluded that aspartase does not play any part in the breakdown of aspartic acid by Aspergillus.

In the same way I investigated whether the keto acids, which should be produced by the oxidative deamination by means of *amino-oxidases*, could be demonstrated. With aspartic acid the keto acid is oxalo-acetic acid, and with alanine pyruvic acid, for both substances there is a very sensitive colour reaction, as worked out by STRAUB (9). To inhibit a further oxidation of the keto acids Na-arsenite was again used; this substance, according to KREBS (l.c.) does not inhibit the deamination itself. It now appeared that, as a product of the deamination of l(—)aspartic acid, oxalo-acetic acid could never be demonstrated, but pyruvic acid was, although in very small quantities. This was also the case with the inhibited oxidation of dl-alanine; it was not investigated in the case of the other amino acids. Pyruvic acid was also produced, however, with the basal metabolism of the Aspergillus mats, when Na-arsenite was added.

According to these experiments it is thus improbable that an amino-oxidase plays any part, since the keto acids would then be produced in larger quantities. As further argument against the action of an oxidase is that the l- and the d-component of aspartic acid and alanine are deaminated exactly alike (Cf. Table 4), whereas for the oxidation of the d-component another enzyme must be active than for the l-component. The d-amino-oxidase is also active in tissue *extracts*, but it appeared that a ground Aspergillus mat was no more able to deaminate the d-component of aspartic acid and alanine than the l-component. Finally it appeared that glycocoll was easily deaminated, whereas this substance is not affected by the amino-oxidases.

The conclusion from the above experiments is, thus, that an amino-oxidase does not occur in Aspergillus niger, or in any case, does not play any part in the amino acid oxidation under the experimental circumstances.

This oxidation is closely connected with the cell-respiration, for with all the methods by which this latter is inhibited or suppressed (addition of Na-arsenite, grinding, working in N_2 atmosphere) the amino acid oxidation is also inhibited, so that the amino acids are probably oxidised with the help of the ordinary cell-respiration enzymes.

That the *glutamic acid dehydrogenase* might play a part in the oxidation of l(+)glutamic acid is very improbable, for a ground fungus mat cannot oxidise this amino acid with methylene blue as H_2 acceptor. Now, it is very well possible that Aspergillus contains the apodehydrase which is activated by the addition of co-dehydrase I or II, as VON EULER et al. (3) found in a number of animal and plant tissues, but this implies nothing after all regarding the action of the enzyme under my experimental conditions, which are very favourable for the fungus used.

Lastly I investigated whether an aminopherase might be active, that is, whether a "transamination" between amino acid and keto acid might take place by the activity of the fungus. These experiments are not yet completed, but no such transamination has been obtained under various conditions indicated as being favourable. As an example, Table 5 gives the result of an experiment with keto-glutaric acid and alanine, from which it appears that neither glutamic acid nor pyruvic acid was formed at all.

TABLE 5. Starved Aspergillus mats on 10 ml phosphate buffer pH 7.4 in N_2 atmosphere. Addition 1 ml 0.2 % Br-acetate solution. Temp. 25°. Concentration in mol. per mat.

Substrate	l (+) alanine	ketoglutaric acid	l (+) alanine + keto-glutaric acid
Initial conc.	0.172	0.228	0.172 + 0.228
Alanine disappeared	0.024	—	0.009
Pyruvic acid formed	0.000	0.000	0.000
Glutamic acid formed	0.043	—	0.000

Summary.

The various enzymes hitherto known for the amino acid breakdown have no part in the deamination of amino acids by starved mats of Aspergillus niger. With this organism the oxidative deamination proceeds best at pH 2—4 of the surrounding solution. It is closely coupled with the ordinary cell-respiration, and is probably caused by "unspecific" oxidation enzymes.

Amsterdam, Laboratory of Plant Physiology
of the University.

LITERATURE.

1. BRAUNSTEIN, A. E. und M. G. KRITZMANN, Enzymologia **2**, 129 (1937).
2. BRAUNSTEIN, A. E., Enzymologia **7**, 25 (1939).
3. EULER, H. VON et al., Z. physiol. Chem. **248**, 227 (1937); **254**, 61 (1938); **255**, 14, 27 (1938).
4. GALE, E. F., Biochem. J. **32**, 1583 (1938); Enzymologia **7**, 21 (1939).
5. GORTER, A., Thesis, Amsterdam, 1940 (in the press).
6. HAHN, A. und W. HAARMANN, Z. f. Biol. **87**, 107 (1928).
7. KREBS, H. A., Ann. Review Biochem. **5**, 247 (1936).
8. MOOI, J. C., Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 195 (1939).
9. STRAUB, F. B., Z. physiol. chem. **244**, 117 (1936).
10. VIRTANEN, A. I. und J. TARNANEN, Biochem. Z. **250**, 193 (1932).
11. WAESBERGHE, H. P. J. M. VAN, Rec. trav. bot. néerl. **34**, 623 (1937).
12. WARBURG. O. und W. CHRISTIAN, Biochem. Z. **295**, 261 (1938).

Biochemistry. — *Tissues of prismatic celloidin cells containing Biocolloids.* **II.** *Coacervation of Gum-Arabic by Toluidin-blue and the phenomena accompanying the dissolution of the coacervate.* By H. G. BUNGENBERG DE JONG and B. KOK. (Communicated by Prof. J. VAN DER HOEVE).

(Communicated at the meeting of April 27, 1940.)

1. *Introduction.*

In an earlier communication [1] we described the accumulation of a basic dye followed by coacervation in the cavities of a collodion membrane containing gum arabic. By the new method of embedding in celloidin membranes, whereby the cells are obtained as tissues [2] this combination was again investigated and extended to the study of the suppression of the coacervate by a very weak salt solution.

2. *Accumulation of Toluidin-blue in the cells containing gum Arabic.*

If a 3 % gum arabic solution be embedded in the celloidin membrane and if, in the cuvette, a greatly diluted Toluidin-blue solution (0.001 %) be allowed to flow over the tissue, in 10 minutes time there will be the beginning of accumulation. The gum arabic solution is coloured violet and this colour will gradually grow deeper.

With respect to the mechanism of the accumulation [3] and the appearance of metachromasia [4], we refer to other publications.

3. *Coacervation in the cell-compartment.*

If 0.01 % Toluidin-blue solution be allowed to flow over the membrane, the accumulation will take place much more rapidly and coacervation very soon follows. In the cells tiny violet-stained drops separate, which at first show a strong Brownian movement. They coalesce gradually into larger drops and sink on to the celloidin wall, which they moisten. After sufficient time of waiting all the coacervate will be found, as metachromatically stained „smear" on the wall of the cell-compartment.

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[1] H. G. BUNGENBERG DE JONG and O. BANK, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 83 (1939).

[2] H. G. BUNGENBERG DE JONG, B. KOK and D. R. KREGER, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 512 (1940).

[3] H. G. BUNGENBERG DE JONG and O. BANK, loc. cit. The theory of the accumulation will be published in more detail in "Protoplasma".

[4] O. BANK and H. G. BUNGENBERG DE JONG, Protoplasma **32**, 489 (1939).

If 0.1 % Toluidin-blue solution be used the same process takes place, only all is much more rapid.

4. *The suppression of the Toluidin-blue-Arabinate-coacervate.*

If, for instance, we obtain coacervation first with 0.1 % Toluidin-blue and wait until the coacervate is entirely spreaded on the wall, and then pour distilled water over the membrane, changes will be seen gradually to take place which indicate that the wetting properties of the coacervate with respect to the wall are changed. Tiny white spots will appear in the smear. These changes will take place much faster if a very weak salt solution (e.g. 5 millimol NaCl) be poured over the membrane. Under these circumstances toluidin-blue will be washed out of the cells, in consequence of which the coacervate will be suppressed after some time, and a stadium results in which toluidin-blue is still accumulated in the cell-compartments. But with further flushing with 5 m. aeq. NaCl the coloration will disappear completely.

The aforesaid phenomena will be comprehensible when one bears in mind that accumulation is less when a neutral salt is present than when it is not.

A constant flow of a weak salt solution over the membrane washes more and more toluidin-blue out of the cells. The first consequence is the suppression of the toluidin-blue arabinate coacervate, after which the diffuse accumulation still left in the cell-compartment disappears. We shall investigate further how the suppression of the coacervate takes place in the cell.

5. *Changes in the wetting properties of the coacervate with respect to the cell-wall.*

The microphoto A represents the final stadium of the coacervation. Whereas it may happen that many cells in the membrane have been injured and many have lost their gum arabic, this is not the case with three cells. In these a large quantity of the coacervate is smeared out more or less evenly on the wall (the white lines are folds in the cell wall). The tissue was now washed with a solution of 5 millimol NaCl per litre. Several consecutive stadia in the changes that occur are shown in the micro-photos B and C. In B we see that the coacervate no longer covers the wall evenly, but has withdrawn materially, wherby rounded masses can be distinguished in the strips still lying on the wall. In C this process has advanced further and there are already coacervate drops in the cell-lumen loosened from the wall (Brownian movement!). After some time there will be no more coacervate on the wall, but merely free coacervate droplets in the lumen of the cells. The morphological changes, as seen in A, B and C, are thus the result of the wetting properties of the coacervate with respect to the wall being greatly altered.

6. *The formation of pulsating vacuoles in the free coacervate droplets.*

Simultaneously with the loosening of the coacervate from the wall, the quantity of coacervate in the cell-compartment gradually diminishes. However the coacervate droplets do not dissolve from outside and thus become gradually smaller in diameter. On the contrary, vacuoles are formed in the coacervate droplets and grow in volume, whereby the droplets even become appreciably larger and transparent in the centre. Many of these vacuolised droplets, some with one, and some with two, vacuoles, can be seen in the micro-photo C. Here we have to do with the same phenomenon as we described previously with respect to the free coacervate droplets, whereby the coacervate gradually dissolves via a stadium with pulsating vacuoles [1].

In the cell-compartment, too, the vacuole embedded in the droplets increases greatly in volume until the peripheral coacervate-skin breaks, and the vacuole-content is discharged, the coacervate-scale contracting again to a tiny ball, the diameter of which is only 1/3—1/5 of what it was before the central vacuole appeared. This process is repeated several times, whereby the remaining coacervate droplet is smaller and smaller after each break, and at length disappears altogether.

That coacervate gradually dissolves in considerable quantities is also seen in micro-photo C from the increasing even staining in the cell-compartments, for, after the complete dissolving of the coacervate, we are left with a stadium in which there is a strong diffuse accumulation of dye in the cell-compartment.

7. *Influence of the electric field during the stadium of the vacuolising free coacervate droplets.*

We have described previously the formation of hollow spheres in the complex-coacervate of gum arabic-gelatine, i.e. coacervate droplets with a very large central vacuole [2]. We assume now, that the formation of hollow spheres and of coacervate droplets with pulsating vacuoles are virtually dependent upon the same mechanism. In both cases water from outside is carried inside through the coacervate, whereby the only difference is that, with the hollow spheres the coacervate-skin does not break open, as it does in the case of droplets with pulsating vacuoles. With regard to a further investigation as to the mechanism of the formation of hollow spheres, we refer to other publications [3], and will merely mention here that their formation was solely dependent upon negative coacervate droplets. Thus, in this connection, it was interesting to inquire whether the toluidin-

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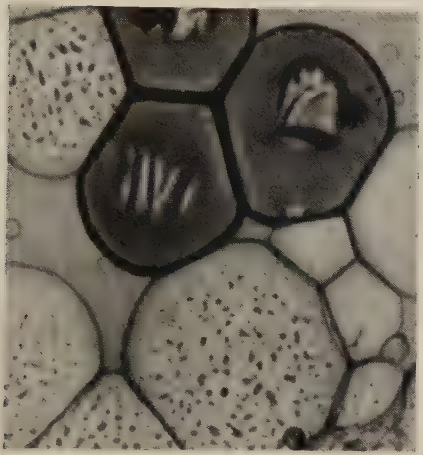
[1] H. G. BUNGENBERG DE JONG, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 41, 643 (1938).

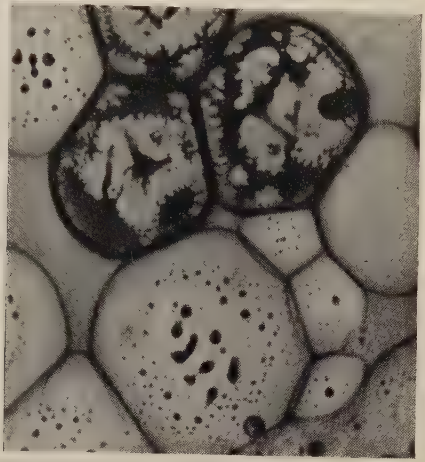
[2] H. G. BUNGENBERG DE JONG and O. BANK, Kon. Ned. Akad. v. Wetensch., Amsterdam, 42, 274 (1939).

[3] Will be published in "Protoplasma".

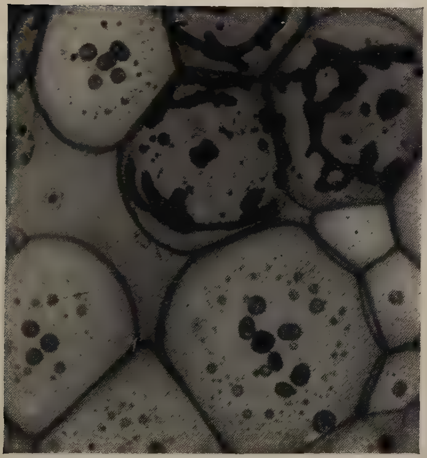
H. G. BUNGENBERG DE JONG and B. KOK: Tissues of prismatic celloidin cells containing biocolloids. II.



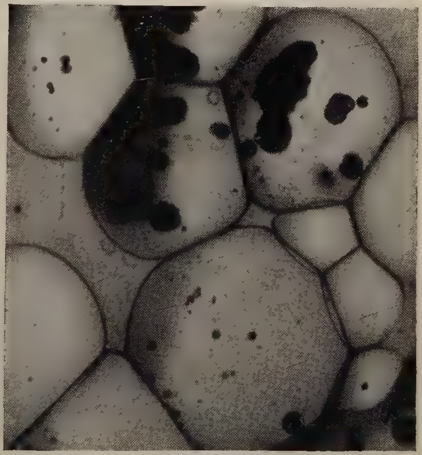
A 190 ×



B 190 ×



C 190 ×



D 190 ×

Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

blue arabinate in the stadium of the vacuolisation, described sub 6, are also electrophoretically negative. For this reason a direct current field of 30 volt/cM was applied shortly after the micro-photo C was taken. Compare the micro-photo D, where the anode lies on the left and the cathode on the right, and where the photograph was taken a few seconds after the field was applied. The vacuolised coacervate droplets move rapidly to the left, i.e. they are negatively charged.

The micro-photo further shows that all the diffuse staining toluidin-blue in the cell-compartment has shifted to the anode.

At first sight this result appears strange for a dye-cation, but we must take into account that the toluidin-blue-cation is not present in a free state, but adsorbed to the negative arabinate.

In conclusion we would remark that one other notable phenomenon was observed indicative of an electric polarisation in the tissue. In switching on the current the coacervate droplets are seen to move rapidly to the anode. If the current be switched off, they move away slightly in the opposite direction. We shall return to this later.

8. *Effect of distilled water.*

In § 4—7 we discussed the suppression of the coacervate with 5 millimol NaCl. If we repeat the same with distilled water, the same phenomena will be observed in principle. Only in this case all takes place much more slowly. Here, too, the liberated coacervate drops enlarge slowly to hollow spheres, whereas it takes very long before they break. The hollow spheres here are also negatively charged and a distinct polarisation takes place in a direct current field.

March 1940. *Laboratory for Medical Chemistry at Leiden.*

Biochemistry. — *Tissues of prismatic celloidin cells containing biocolloids.* **III.** *a. Behaviour of an embedded complex-coacervate in the electric field. b. Polarisation phenomena.* By H. G. BUNGENBERG DE JONG and D. R. KREGER. (Communicated by Prof. J. VAN DER HOEVE).

(Communicated at the meeting of April 27, 1940.)

1. *Introduction and Methods.*

The method described in a previous communication [1]) for embedding biocolloids in celloidin membranes enables us not only to investigate the coacervation phenomena dependent upon an altered milieu, but also to study the influence of the electric field upon the resulting coacervates.

For this purpose the membranes, spread on a water-surface and fixed on a cuvette, as has been fully described in the earlier communication above-mentioned, were exposed to the influences to be studied and examined under the microscope.

This cuvette has been arranged so that the object-glass can be laid on it with the membrane downwards, and any selected fluid allowed to flow over it and, at the same time, an electric current may be applied. The flushing fluid flows off along the electrodes, so that the electrolysis products are carried away. Fluid and cuvette are so heated that the temperature of the membrane is never lower than 30°.

As objects of study the complex coacervates of gelatine and gum arabic were selected, for reason that this combination, which has been studied exhaustively before [2]), rapidly forms parietal coacervates in the celloidin cells. The stock solution embedded in the tissue was always the same, viz. 6 gr. gum arabic + 5 gr. gelatine + 200 cc distilled water.

In this the coacervation was brought about at different pH, with or without 10 m. aeq. KCl. The pH variation was obtained by allowing acetic acid of different concentrations to flow through the cuvette. Reversal of charge of the complex coacervate at the selected proportion of gelatine and gum arabic is near pH 3.35 (0.01 N acetic acid). The coacervation can be attained at all pH values between pH 4 (0.0005 N acetic acid) and PH 2.65 (0.2 N acetic acid) [3]).

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[1]) See Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, XLIII, 512 (1940).

[2]) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Kolloid Beihefte **43**, 143, 213 (1936).

[3]) Cathaphoretic determination at 36° in a microscopical cuvette yielded pH 3.32 as reversal of charge of the free coacervate drops, and pH 3.28 with 10 m. aeq. KCl.

The influence of the electric field was examined on negative, uncharged as well as on positive coacervates.

For the strength of the direct current field 30 V/cm was taken, as a rule, if there was no KCl in the acetic acid solution used to flush the membrane, and 10 V/cM when 10 m. aeq. KCl was present. The salt accélerates the phenomena very materially, so that, with 10 V/cM and KCl, the process is still appreciably more rapid than with 30 V/cM without the addition of salt.

The phenomena in the electric field were studied on ± 100 membranes. As a rule a new membrane was prepared for each experiment.

The description which here follows of these phenomena has been made as concise as possible, and will be published in more detail at another place (in "Protoplasm").

2. *Phenomena in the electric field.*

These are:

1. If we look through the microscope immediately after the object-glass has been laid on the cuvette, it will be possible, as was earlier described in detail, to see the coacervation taking place.

A current of short duration will show that the droplets and vacuoles in process of formation behave in accordance with the sign of charge of the coacervate, as this is determined by the pH of the flushing acetic acid, i.e. in the case of negative coacervates they migrate to the anode and in the case of positive coacervates to the cathode.

2. When the coacervate has completly formed and has taken up its topographical ultimate position (Fig. 1a), the current being applied, vacuolisation takes place in the parietal coacervate both right and left of the vacuole (Fig. 1 *b*). (See microphoto A.)

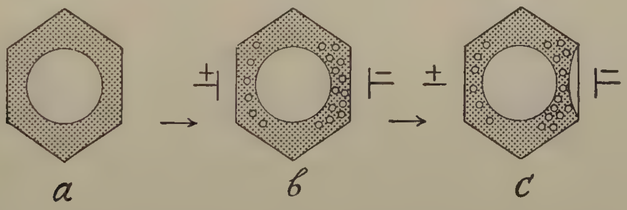


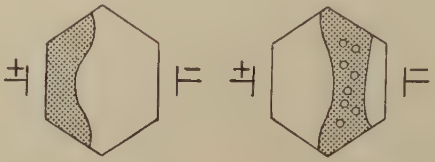
Fig. 1.

The vacuolisation in the coacervate right of the vacuole ("right" is here always the cathode side) is invariably much more marked than that to the left of the vacuole, while, moreover, at the right wall of the cell a secondary vacuole is formed, which expounds to the left (Fig. 1c). This always takes place at the cathode side, irrespective of the sign of charge of the coacervate. See microphoto B.

3. After a short time (± one minute, dependent on the voltage and

presence of KCl), we see in most of the cells containing the positive coacervate, the vacuole on the right and the coacervate on the left (Fig. 2a).

In the case of the negative coacervate, then in most cells the coacervate will be on the right and the vacuole on the left. In this "right" coacervate the vacuolisation described above and the formation of the secondary vacuole continues to take place (Fig. 2b).



Coac. positive Coac. negative
a Fig. 2. b

4. Again, after a short time (± 3 minutes according to the voltage, etc.) a difference will be noticed between the right and left halves of the cell-groups.

Then, in the positive as well as in the negative coacervate, we see in the left half of the cell-groups the coacervate at the right in the cell (Fig. 3a), whereas in the right half of the cell-groups, it will generally be to the left in the cell (Fig. 3 c). See microphoto C. The coacervate standing at the right side (cathode) in the cells (Fig. 3 a) continues vacuolising, especially if KCl is present; the coacervate on the left side (Fig. 3 c) does not do so and is frequently almost vacuole-free.

In the larger cells in the middle of the cell-groups a condition now arises, whereby the coacervate divides the cell, like a partition, into two parts, or stands like two heaps opposite each other. These heaps are often connected by thin coacervate strips along the upper and lower wall of the cell. In Figure 3 b these conditions are seen.

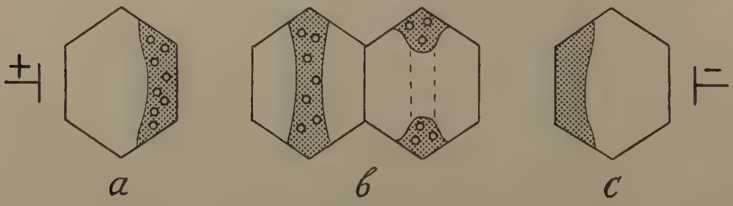
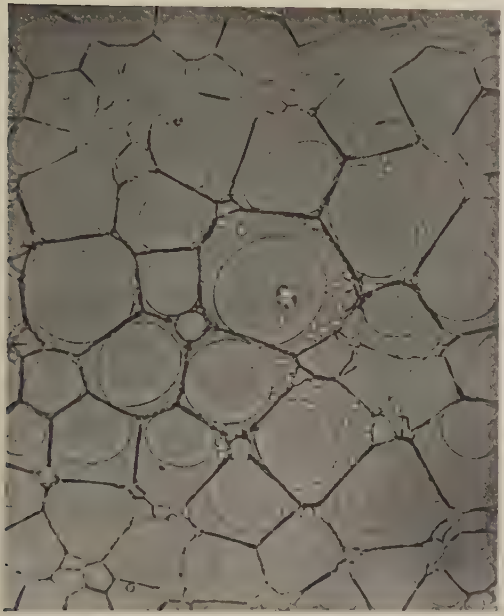


Fig. 3.

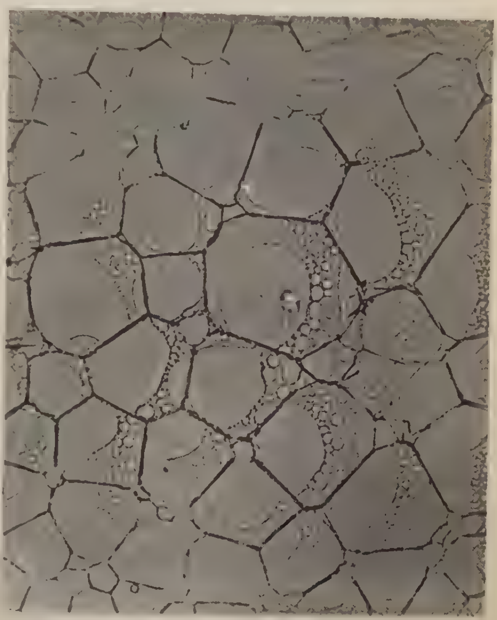
It may be mentioned as a peculiarity that the coacervate partitions are a conspicuous feature of nearly the entire cell-group when the polarity is reversed during the process.

After this, if the current be continued, no fundamental changes take place, except that the coacervate in the cells becomes quantitavely less and less.

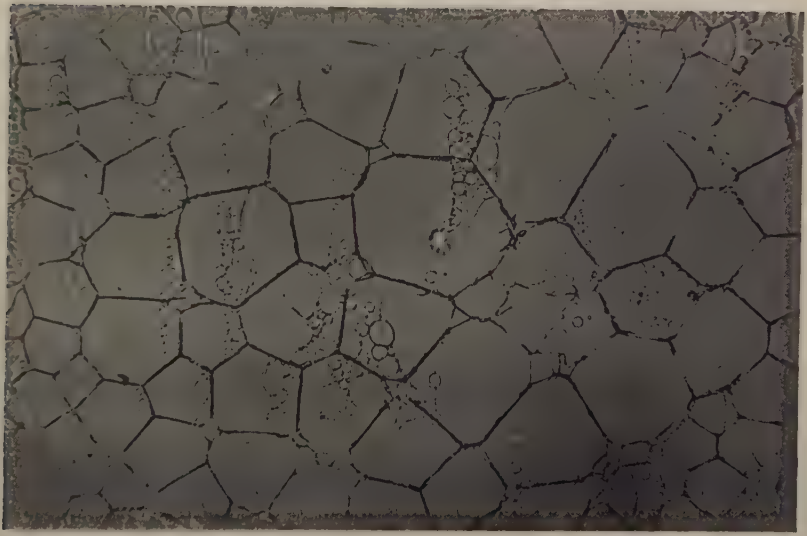
H. G. BUNGENBERG DE JONG and D. R. KREGER: Tissues of
PRISMATIC CELLOIDIN CELLS CONTAINING BIOCOLLOIDS. III.



A



B



C

Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

3. *Differentiation of the phenomena in the electric field into two groups.*

We have seen in § 2 that the action of the electric field leads to an ultimate condition in which, practically speaking, no further changes occur and in which only the slightest difference, or none at all, between positive and negative coacervate is to be observed. In the course of the process, however, there do appear phenomena which vary with the sign of charge of the coacervate (e.g. the movement of the coacervate droplets and vacuoles during the coacervation process and the position of the coacervate and vacuole with respect to each other in Fig. 2). The phenomena which, in the case of free complex coacervate droplets, occur in the electric field were dealt with fully on a former occasion 1)). It then appeared that in the positive coacervate these are the reverse of those in the negative. Some of the phenomena seen in § 2 may be attributed to this (e.g. the shifting of the vacuoles).

The other phenomena in § 2, and which are the most noticeable ones, are never observed in the case of free coacervate droplets. In this respect it is typical that they are independent of the sign of charge of the coacervate, so that they must be ascribed to the presence of the surrounding celloidin walls.

4. *Working hypothesis concerning the part played by the celloidin walls.*

If we assume that in the electric field concentration differences of the ions, in particular of the H ions, arise between either side of the celloidin walls of the cell-compartiments 2)), the principal phenomena (i.e. those whereby the sign of charge of the coacervate has no part) can be explained.

It is obvious that we may assume that in consequence of the said polarisation, the pH will rise on the anode side of the celloidin membrane but fall on the cathode side (confirmation of this hypothesis will be found in the following paragraphs).

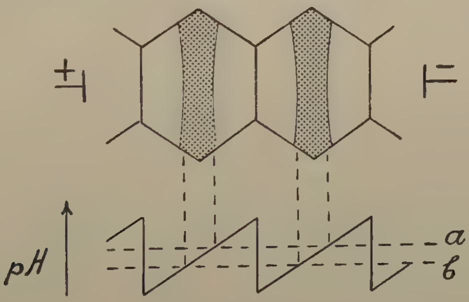


Fig. 4.

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1) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, loc. cit. see p. 201—203. See also H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Biochem. Z. **221**, 403 (1930).

2) BETHE and TOROPOFF, Z. physikal. Chem. **39**, 613 (1902).

Considering some contiguous cells, we might expect in every cell-compartment the pH, consequently, to rise from left to right (see Fig. 4).

In connection with the fact that, with a given ratio of gum arabic and gelatine, the complex coacervation is only possible in a certain pH-range (in Fig. 4 between the lines a and b), the typical position of the coacervate in the middle cells of a group during the ultimate stationary condition (Fig. 3b) is comprehensible.

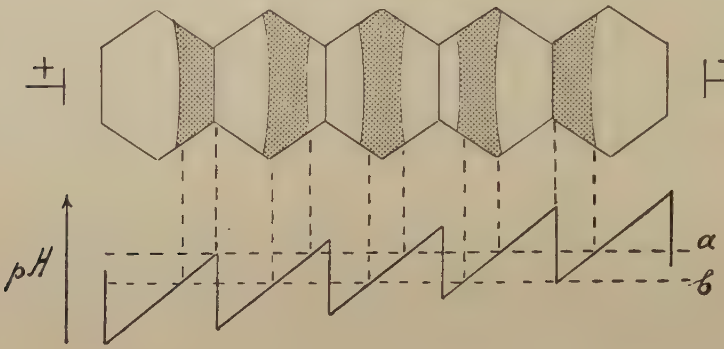

Fig. 5.

In these cells the pH is then too low in the left half, and too high in the right half, and the coacervate can, therefore, exist only in an intermediate zone perpendicular to the field.

If, in the final stationary stadium, we regard, however, not merely the middle cells of a cell-group, but also the cells on either side, we shall come to the conclusion that the average pH in the cells rises from left to right in one cellgroup (left anode, right cathode).

For we notice that in the left half of a cell-group the coacervate is no longer situated in the middle, but at the right side-wall, and similarly that in the right half of a cell-group the coacervate is just at the left side. See fig. 3*a*, 3*c* and microphoto C.

The pH range in which the coacervate can exist is, thus, in the left cells quite at the right side of the cell, i.e. the average pH is lower here than in the middle cells. Inversely, in the right cells the pH range in which there can be a coacervate, is just quite at the left in the cells, i.e. the average pH is higher here than in the middle cells (see Fig. 5).

We cannot discuss here further the other details of § 2, but will merely state that their explanation may advantageously be based upon the two hypotheses discussed above, viz. in consequence of the polarisation of the celloidin walls there develops a) a pH gradient inside each cell, and b) a gradient of the average pH over the whole cell-group.

In the following paragraphs an account will be given of experiments which confirm both these hypotheses.

5. *Macromodel for the origin of pH differences on both sides of a membrane after applying an electric field over the membrane.*

For this a U-shaped tube was used, one arm of which is divided by two perforated plate-glass slides lying one upon the other (see Fig. 6). The upper one is cemented to the upper half of the tube, the lower to the lower half. When a cellophane membrane has been inserted between the slides, the upper part of the tube can be laid loosely upon the lower half.

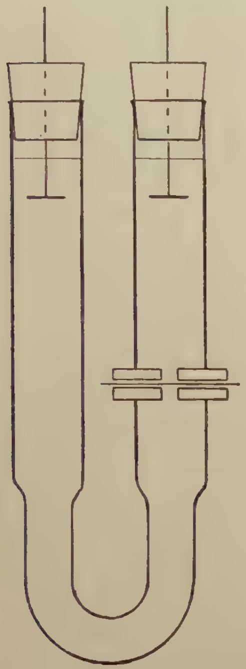
Fig. 6.

The cuvette is filled with a solution of bromthymol-blue in distilled water ¹) up to the upper edge of the slide, the piece of cellophane is laid upon this slide, the upper part then laid down upon it, and the two arms of the tube filled up equally, whereupon 2 electrodes are put into each. To get the potential gradient over the membrane as large as possible, the arms of the cuvette are taken very wide, ± 3.5 cM, and the bore in the slides not more than ± 7 mm in diameter. On the vertical outside edge of both plate-glasses, a spot was made thoroughly transparent, so that any staining in the columns of liquid on either side of the membrane could be observed by looking through the plate-glasses lengthways.

When the current was applied (± 60 V), blue staining appeared in the brownish-yellow fluid on the anode side of the membrane. The same result was obtained with 10 m. aeq. KCl.

This experiment shows that the shifting of the pH in the electric field, as was assumed above in § 4, (the pH rises on the anode side of the membrane) does actually take place, that is to say, with a cellophane membrane. In the following paragraphs we shall see that the same is the case with the membranes of the microscopic cells in the celloidin tissue.

6. *Demonstration of the pH gradient inside the cells and over the entire cell-group by means of bromphenol-blue.*

If the pH differences occurring left and right in the cell, as well as in the cell-group are indeed the cause of the principal phenomena described in § 2, they must be so great that they can be shown with indicators. The difficulty here is to get an indicator with a suitable range in such a high concentration in the cell that this very thin layer of fluid, no thicker than the height of one cell, will allow an observable colour change.

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¹) The stock solution: 0.2 gr. bromthymol-blue + 3.2 cc NaOH o.1 N dissolved in 500 cc. distilled water, was diluted five times with distilled water.

No results are to be expected unless an indicator is used which, in one of its forms, is accumulated (adsorbed) in the complex-coacervate gum-arabic-gelatine.

For our purpose bromphenol-blue (pH 3.0 yellow, pH 4.6 blue) proved suitable. From a bromphenol-blue solution, even if pH is such as to have an intermediate shade, or a more pronounced blue colour, the yellow form is nevertheless always adsorbed by the complex-coacervate. The yellow form is the non-dissociated form, and the binding must be looked for here in the general affinity of phenol-groups with proteins. This last results in dehydration, and if we follow the behaviour of the complex-coacervate in the cell-membrane in the electric field, after the yellow staining with bromphenol-blue, we shall also see distinct phenomena which indicate a less aqueous, and thereby a more viscous, state of the coacervate. For this experiment the celloidin membrane was flushed with a mixture of 80 vol. % 0.01 N acetic acid + 20 vol. % bromphenol-blue stock solution [1]). When, after 5 minutes a plainly observable staining is seen (light yellowish-green if looked at microscopically in bluelight (clear blue sky) the experiment proper can be proceeded with, either at once, or after a short rinsing of the cuvette with 80 vol. % 0.01N acetic acid + 20 vol. % distilled water.

When the current was applied there was, according to the rule, a strong vacuolisation on the cathode side. The expounding of the secondary vacuoles to the left was very slow and the coacervate made the impression of being very viscous. After some time partitions of green coacervate appeared in nearly all the cells of a shape which differed in some respects from the coacervate partitions, as these have been described above.

About 5 minutes after the current was applied the vacuole fluid in the right half (cathode side) of the cell began to stain blue in some places, especially in the right cell-groups.

In a short time most of the cells in the right cell-group-halves on the cathode side were coloured distinctly blue. This was the case hardly anywhere in the left cell-group halves. The coacervate generally in the middle of the cells remained green, which, especially at the left side of the partitions and in the cells at the left (anode side) of a cell-group became more a yellowish-green.

Herewith the pH gradient, a) inside the cell and, b) over the whole cell-group, has been demonstrated.

7. *Demonstration of the pH gradient inside the cells and over the entire cell-group by means of resorcin.*

As we stated above, phenols exercise in general a dehydrating effect upon proteins. If the charge of the protein is small, phenols cause flocculation or coacervation. Also with gelatinsol we find that resorcin brings about coacervation in a fairly limited pH range round about the I.E.P.

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[1]) 0.2 gr. bromidephenol-blue + 2 cc NaOH 0.1 N dissolved in 500 cc distilled water.

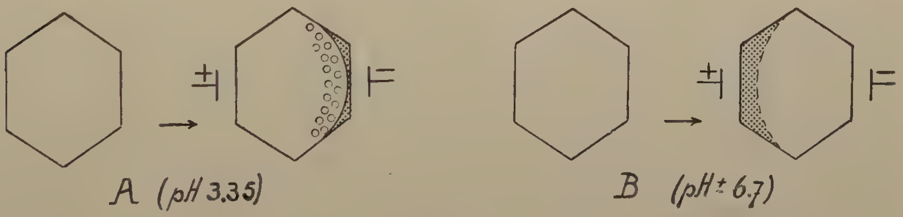
We may make use of this to inquire, in another way than with the aid of indicators, whether the assumed pH shifts actually occur in the electric field.

If gelatine only be enbedded in the membrane and it be flushed with a resorcin solution with a pH which lies outside the pH range in which the coacervation takes place, it should be possible that, in applying the field, the pH on *one* side of the cells is raised or lowered to such a degree that it comes within the coacervation area and we shall see the coacervation take place here.

In consequence of the change of the average cell pH from left to right in a cell-group, this phenomenon will be more distinct again in the one half of the group of cells than in the other.

Such a condition can indeed be realised. For this a 5 % gelatine solution in distilled water is embedded in the membrane.

This was then flushed with a 0.01 N acetic acid solution (pH 3.35), and this solution was followed, without any break, by an acetic acid solution of the same concentration in which, however, 4 % resorcin had been dissolved. This solution was passed through the cuvette for 10 minutes, after which there was not yet any coacervation to be seen in the cells. When the current was applied the coacervation appeared in most of the cells in numberless tiny droplets against the wall on the right side (see Fig. 7 A). In the right half of the cell-group the coacervation extended farther towards the middle of the cells than on the left side, whereas in many of the cells quite to the left in the cell-group there was no coacervation.The pH of the wash was lower than the coacervation range. As has been exposed above, the pH will become higher on the anode side of the membranes, on the cathode side lower, when the current is applied. The appearance of the coacervate in the cells on the right side is thus in accordance with this.



A (pH 3.35)
B (pH ± 6.7)

Fig. 7.

Now, taking a medium which has a pH lying above the coacervation area, we should, in accordance with the same reasoning as above, be able, when the current is applied, to bring about coacervation in the left cell-half, since pH is here lowered.

A pH of ± 7 proved to be the best suited for this, since a buffer solution had already been tried in the preceding experiment to attain a certain pH and had yielded no coacervation, possible owing to the eliminating of the pH differences by the buffer; this pH was obtained by neutralising the acid reaction of the 4 % resorcin solution (per 100 cc 10 cc NaOH 0.1 N was added). With this solution bromthymol-blue was greenish-blue.

This solution was washed for 10 minutes over the membrane with embedded gelatine, after which still no coacervation occurred. When the field was applied, coacervation very quickly occurred (± 20 sec.) as was to be expected, at the left of the cells (Fig. 7 B). Moreover, the difference between anode and cathode sides of a cell-group was again observed, namely on the anode (left) side the phenomenon was stronger than on the right side, so that a change of the average pH for the whole cell-group was evinced here in accordance with the hypothesis in § 4.

When the current was cut off the coacervate disappeared within a few seconds.

March 1940. *Laboratorium for Medical Chemistry at Leiden.*

Anatomy. — *On the cephalization stage of Pithecanthropus erectus and Sinanthropus pekinensis.* By R. BRUMMELKAMP. (Communicated by Prof. C. U. ARIËNS KAPPERS).

(Communicated at the meeting of May 25, 1940.)

There is a difference of opinion among the anthropologists of to-day as to the relation between Pithecanthropus erectus and Sinanthropus pekinensis and the place they occupy in the succession of primates. According to WEIDENREICH [1]) Sinanthropus is more primitive than Pithecanthropus, but this difference is not such that the discussion of the degree of primitiveness of Pithecanthropus and Sinanthropus must be considered of great importance. When compared with other fossil hominids, there is, according to WEIDENREICH, no doubt that *Pithecanthropus and Sinanthropus represent the same general stage of evolution*, the difference existing between the two types having to be considered as regional or racial deviations. A similar view is expressed by BOULE [2]) when he says that the two organisms are very intimately related, if not identical. ARIËNS KAPPERS and BOUMAN [3]), ARIËNS KAPPERS [4]), DUBOIS [5]), KOENIGSWALD a.o. [1]) hold a different opinion. On the ground of the fissuration of brain-casts, general contours, brain capacity and other anatomical details, they believe that *Pithecanthropus erectus and Sinanthropus pekinensis should not be considered as belonging to one species,* but they think that Sinanthropus is nearer to the more advanced Javanthropus soloensis Oppenoorth, or to the Rhodesian-man, while assigning to Pithecanthropus a separate, more primitive place in the succession of primates.

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[1]) WEIDENREICH, FRANZ. Pithecanthropus and Sinanthropus, Nature, Vol. 141, p. 378—379, 1938; Discovery of the Femur and the Humerus of Sinanthropus pekinensis, Nature, Vol. 141, p. 614—617, 1938; The relation of Sinanthropus pekinensis to Pithecanthropus, Javanthropus and Rhodesian Man, Journal of the Royal Anthropological Institute, Vol. LXVII, 1937; The new discoveries of Sinanthropus pekinensis and their bearing on the Sinanthropus and Pithecanthropus problems, Bulletin of the Geological Society of China, Vol. XVI, 1936—1937; Observations on the form and proportions of the endocranial casts of Sinanthropus pekinensis, other hominids and the great apes: a comparative study, Palaeontologica Sinica, Series D, Vol. VII, fasc. 4, 1936.

[2]) BOULE Marcellin. Les fossiles, Paris 1935.

[3]) ARIËNS KAPPERS C. U. and K. H. BOUMAN. Comparison of the endocranial casts of the Pithecanthropus erectus skull found by DUBOIS and VON KOENIGSWALD's Pithecanthropus skull, Kon. Ned. Ak. v. Wet., Proceedings Vol. XLII, No. 1, 1939.

[4]) ARIËNS KAPPERS C. U. De groeven op de lobi frontales van Pithecanthropus erectus DUBOIS, vergeleken met die van den Neanderthalmensch, Homo recens en Chimpanzee, Kon. Ned. Ak. v. Wet. Meded. Dl. XXXVII, No. 10, 1928.

[5]) DUBOIS, EUGÈNE. On the fossil human skull recently described and attributed to Pithecanthropus erectus by G. H. R. VON KOENIGSWALD. Kon. Ak. v. Wet., Proceedings Vol. XLI, No. 4, 1938; Die phylogenetische Grosshirnzunahme autonome Vervollkommnung der animalen Funktionen, Biologia generalis, Bd. VI, pp. 247—292, Wien 1930; On the principal characters of the femur of Pithecanthropus erectus, Kon. Ak. v. Wet., Proceedings Vol. 29, 1926; Racial identity of Homo soloensis Oppenoorth (including Homo modjokertensis VON KOENIGSWALD) and Sinanthropus pekinensis DAVIDSON BLACK. Kon. Ak. v. Wet., Proceedings Vol. 39, No. 10, 1936.

Thus far no use has been made for the above problem, of the *cephalization-rule*. Yet it seems useful to me to pay some attention to this rule, because it establishes a relation between brainweight c.q. skull capacity on the one hand, and body size (expressed in bodyweight, stemlength or certain femur-measurements) on the other, which relation is of the greatest importance for a right appreciation of the significance, which in a certain case is to be ascribed to brainweight, c.q. skull capacity.

Since DUBOIS' publications we know that there is a relation between brainweight E and bodyweight P, according to $E = c.P^r$. When we apply this rule to nearly related animal species, e.g. cat and tiger, it appears that the value of the co-factor c is constant and that of the exponent $r = {}^5/_9$. When this rule is applied to species which are not so nearly related, though belonging to the same order, we find the remarkable fact that the value of the co-factor varies leapwisely, approximately according to the terms of a geometrical progression, whose ratio is $\sqrt{2}$, parallel to the general level of development of the animals in the order examined [2]).

Hence the value of the co-factor expressing the degree of development of the cerebrum in so far as it depends on body size, is also an index of the stage in evolution of the animal species concerned.

This co-factor is generally called the cephalization coefficient, and the stage in evolution expressed by it is the cephalization stage or cephalization level.

> Thus we see, for instance, that in the order of the *Ungulates*, of the primitive Tragulus the cephalization coefficient is half, of the domesticated forms of Bovis $\sqrt{2}$ times smaller, of the undomesticated Cervidae $\sqrt{2}$ times greater, and of the highly developed Elephas $2\sqrt{2}$ times greater than that of the majority of animal species belonging to this order. In the same way it appears in the order of *Primates* that in humans the value of factor c is 4 times greater than in Gorilla and Simia satyrus and $2\sqrt{2}$ times greater than in Chimpanzee.

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[1]) VON KOENIGSWALD G. H. R. Ein neuer Pithecanthropus-Schädel. Kon. Ned. Ak. v. Wet., Proceedings Vol. 41, 1938.

[2]) Extensive data concerning the leapwise variation of the co-factor c (cephalization coefficient) are found in a.o. BRUMMELKAMP, Normale en abnormale hersengroei in verband met de cephalisatieleer, N.V. N.H. Uitg. Mij (1937), where an attempt is made to account for this leapwise variation. The probability is set forth of a connection with the leapwise development of the cortex cell nuclei, which is according to a $\sqrt{2}$ fold.

When the cephalization rule is applied to the various human races, the value of the exponent is $^{5}/_{3}$ and that of the co-factor is constant, when the *length of the body stem S* (length of head and trunk) is taken as measurement of the body size. The above general formula then becomes $E = c.S^{5/3}$. Written in its logarithmic form, it is $lg\, E = lg\, c + {}^{5}/_{3}\, lgS$, i.e. of the first degree and hence may be graphically represented by a straight line. If we put out $lg\, E$ against $lg\, S$ of a number of human races, it appears, according to fig. 1 that indeed the different relation points of these races are on or near a straight line with a $^{5}/_{3}$ direction, and so answer to our formula. From this it also appears that, according to the cephalization measure, the stage in evolution of the various human races is the same. The data used in fig. 1 are found in table I.

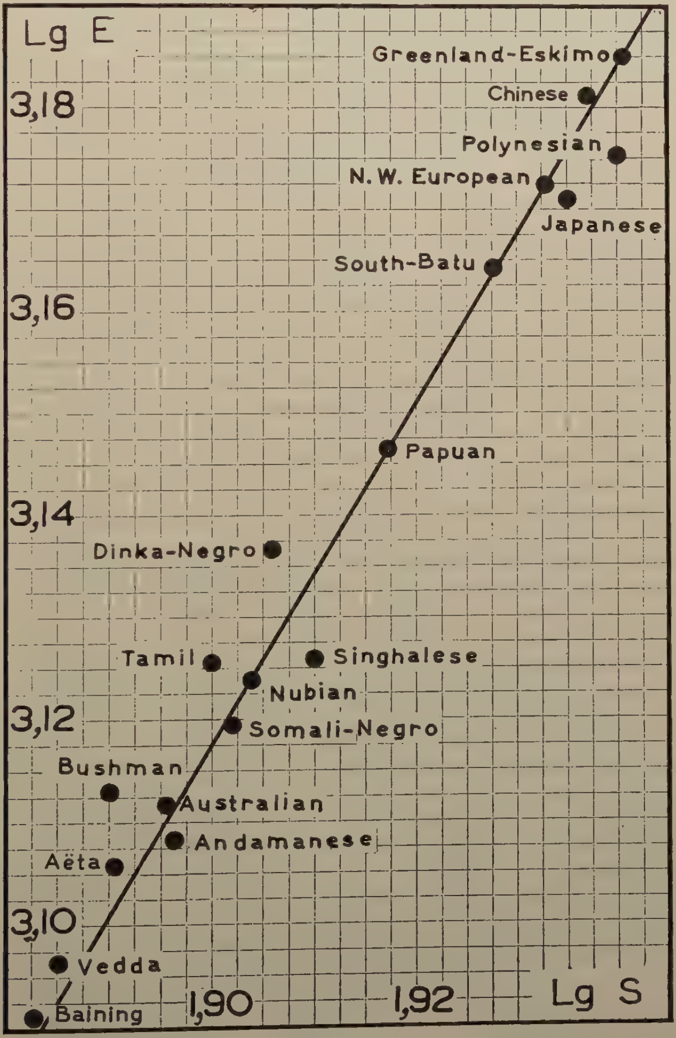

Fig. 1.

TABLE I[1]).

Name	Bodylength	Stemlength (S)	Skullcapacity (E)
N. W. European	171.5	85.75	1490
Vedda	153.3	76.65	1250
Singhalese	162.5	81.25	1336
Polynesian	174.0	87.00	1500
Papuan	165.7	82.85	1398
South Batu	169.6	84.80	1460
Bushman	155.2	77.60	1297
Baining	152.9	76.45	1232
Chinese	160.6	86.56	1518
Japanese	159.3	86.02	1485
Eskimo	162.1	87.53	1528
Andamanese	148.6	78.76	1281
Aeta	146.3	77.54	1275
Dinka Negro	173.0	80.45	1389
Somali Negro	171.5	79.75	1315
Australian	165.8	78.73	1295
Nubian	167.0	80.16	1329
Tamil	165.3	79.34	1336

The relation between skull-capacity c.q. brainweight E and stemlength S will prove valuable in calculating the cephalization of fossil hominids and prehominids. In order to make this clear the following:

When we compare two extreme races, e.g. Esquimaux and Australians, we note that in Esquimaux a great skull capacity is attended with a long body stem, and that also their head and trunk exceed those of Australians in length and transverse measurements, and are therefore heavier, which greater weight is carried by shorter, more robust femora. Skull capacity on the one hand, and length of the body stem, weight of trunk and head and carrying capacity of the femora on the other, apparently vary simultaneously in the same direction.

As the carrying capacity of the femur is proportionate to the load to be carried, i.e. the weight of head and trunk, and as this weight is proportionate to the third power of the stemlength, the skull capacity, which in the various races is proportionate to the $^5/_3$ power of the stemlength, will be proportionate to the $^5/_9$ power of the carrying capacity of the femora. Presuming that there is a close relation between shape and function of the femur, we may expect a relation between skull capacity and carrying capacity of the femur as expressed in the measurements of this bone. It is probable that the carrying capacity is proportionate to a certain power of the area of the transverse section in the middle of the femurshaft, and inversely proportionate to a certain power of the femur length. As length we take the maximal length L and as area of the transverse section in the middle of the diaphysis the product of the sagittal and transverse diameter $(d_1 \, . \, d_2)$. When, for some reason, the area of the transverse section of the diaphysis middle cannot be applied, we may substitute the quotient of the femur volume (V) divided by the quadrate of the length (L^2). Thus we have at our disposal two formulas, *viz.* $E = f\,(d_1 \, d_2/L)$ and $E = f_1(V/L^2)$. On the ground of what has gone before we expect that the two relations will answer to the general form $Y = c.X^r$.

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[1]) For these data about bodylength, stemlength and skullcapacity, I refer to my book: Normale en abnormale hersengroei in verband met de cephalisatie-leer, Uitg. N.V. Noord-Hollandsche Uitg. Mij, 1937.

In order to check the correctness of this presumption we set out lgE against $lg\,(d_1 \, . \, d_2/L)$ and $lg\,(V/L^2)$ respectively.

In fig. 2 we have set out $lg\ E$ and $lg\ (d_1 \, . \, d_2/L)$ of 30 representatives of greatly differing recent and fossil human races. This graph shows that the points thus constructed are approximately in a straight line, which has a $^5/_9$ direction. (In exact figures, calculated by the method of the smallest quadrates, the direction of the line is 0.5403.) The value of the co-factor is 1084, so that this new relation is

$$E = 1084\,(d_1 \, . \, d_2/L)^{5/9} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(I)}$$

By introducing quotient V/L^2 instead of $d_1 \, . \, d_2/L$ we obtain fig. 3. It is seen that again the relation-points lie near a line with a $^5/_9$ direction. The value of the co-factor is 38508, so the second relation is

$$E = 38508\,(V/L^2)^{5/9} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(II)}$$

The data applied in the two graphs are found in tables II and III. *The graphs show that the cephalization stage of fossil hominids does not differ from that of recent hominids.*

We will now use formulas I and II to determine the cephalization stage of Pithecanthropus and Sinanthropus.

Of *Pithecanthropus erectus* both cranium and femur are known. On the ground of the cranium DUBOIS estimates the total capacity of the Pithecanthropus skull at 908 cc "by comparing the remains of the cranial cavity with the corresponding part of the cranial cavities of monkeys which are likest to it as far as their main proportions are concerned (Hylobates, Gorilla, Pongo, Pygathrix, Macacus), taking into consideration the deviations owing to the different size-relations between brain cavity and orbit."

TABLE II.1)

Name	Femur					Skullcap. (E) (measured)	Skullcap. (calc.)	Error %
	max. length (L)	circumf.	d_1	d_2	$d_{av.}$			
Bavarian, m.	445	88	—	—	28	1490	1485	0.25
American, m.	460.81	—	28.36	29.35	—	1490	1506	1.07
Norvegian, m.	461.8	90.7	—	—	28.87	1490	1505	1.07
Norvegian, f.	417.4	79.7	—	—	25.36	1340	1378	2.84
Eskimo (Sm. S.), m.	397.25	—	24.5	31.5	—	1570	1568	0.13
Japanese, m.	410.7	84.1	—	—	26.77	1485	1477	0.54
Japanese, f.	382.7	72.6	—	—	23.11	1319	1305	1.01
Corean, f.	414	77	—	—	24.51	1345	1333	0.89
Paucarcancha, m.	397.5	75	—	—	23.87	1349.2	1324	1.87
Paucarcancha, f.	371.34	68.2	—	—	21.71	1197.4	1236	3.29
Patallacta, m.	401.66	77.3	—	—	24.6	1401	1361	2.86
Patallacta, f.	370.76	67.5	—	—	21.49	1207.3	1225	1.47
Torontoy, m.	427.8	79.5	—	—	25.31	1357.7	1357	0.05
Torontoy, f.	382.3	68.8	—	—	21.90	1214.4	1230	1.28
Ona-Indian, m.	434.8	85.5	—	—	27.23	1487	1458	1.95
Yaghan-Indian, m.	415.5	84.5	—	—	26.9	1420	1475	3.87
Negro, m.	455	83.7	—	—	26.64	1389	1388	0.08
New-Caledonian, m.	439.6	—	24.75	29.9	—	1440	1448	0.56
New-Caledonian, f.	417.5	—	22.5	26.3	—	1315	1316	0.08
Loyalty-Islander, m.	438.4	—	24.7	29.07	—	1463	1426	2.53
Loyalty-Islander, f.	414.7	—	22.5	26.6	—	1326	1329	0.23
Lapp, m.	412.8	82.8	—	—	26.42	1465	1451	0.96
Lapp, f.	384.2	75.4	—	—	24.0	1305	1357	3.98
Fuegian, m.	414.3	88	—	—	28	1590	1545	2.83
Pěsěchěm, m.	406.5	—	22.25	24.75	—	1299	1283	1.23
Vedda, m.	440.6	—	24.1	25.09	—	1293	1292	0.08
Spy I, m.	420	90	—	—	28.65	1562	1573	0.70
Chapelle aux Snts, m.	430	—	29	31	—	1626	1633	0.43
Combe-Capelle, m.	425	—	25	28.5	—	1435	1445	0.70
Grimaldi, f.	436	—	24.4	25.4	—	1315.5	1318	0.19
Chancelade, m.	408	—	27.5	30.0	—	1595	1603	0.50

1) Notes to Table II.

Bavarian. Data of skullcapacity (*SC*) and femurmeasurements in MARTIN (1928). *American*. Data of femurm. in N. WILLIAMS INGALLS (1924). He gives the average values of 200 femora of adult men; the value of *SC* equals that of the average N.W. European. *Norwegian*. Data of femurm. in SCHREINER (1935), who gives the data of 521 male and 493 female adult femur-specimens. *SC* as in N.W. European. *Smith-Sound-Eskimo*. Data of femurm. and *SC* taken from the description of an adult male skeleton by HRDLIČKA (1910). *Japanese*. Data of femurm. in FAYUN NASHIZUKA (1926) who describes 10 male and 10 female adult skeletons, but does not mention *SC*. *SC* taken from ADACHI (see MARTIN, 1928). *Corean*. Description of complete skeleton and skull of adult robust woman by KOGANEI (1906). *Peruvian Highlanders*, described by GEORGE GRANT MCCURDY (1926). *Paucarcancha*, male 60 skulls, 51 femora; female 50 skulls, 44 femora; *Patalacta*, male 24 skulls, 18 femora; female 18 skulls, 13 femora; *Torontoy*, male 11 skulls, 15 femora; female 9 skulls, 9 femora. *Indian*. J. VILK HULTKRANZ (1900) describes skulls and skeletons of 3 *Ona Indians* and 2 *Yaghan Indians*. *Negroes*. Data of femurm. taken from DUBOIS (1926) referring to 26 specimens. Average skullcapacity from the values given by DRONTSCHILOW (1913), ISERLIN (1914) and BENNET BEAN (1906); the latter calculated from brainweight. *New-Caledonian*. SARASIN (1922) gives the average values of 34 male and 27 female skulls and the average values of 11 male and 8 female femora (adult). *Lapp*. SCHREINER (1935) gives the average values of 303 adult male femora and *SC* of 157 adult male skulls; besides of 262 adult female femora and 133 adult female skulls. *Fuegian*. MARTIN (1894) describes 21 skeletons and gives the average *SC* of 25 skulls of adult males. The measurements of the middle of the diaphysis are not given, but "essentially they are not different from ours (European)" (MARTIN). Therefore I used the circumference of the average Bavarian femurdiaphysis. *Pĕsĕchĕm*. Description of a male skeleton by VAN DEN BROEK (1918). On account of the exceptional pilastriform shape of the femurdiaphysis I took the average value of the upper, middle and lower sagittal measurements for the sagittal diameter in the middle of the diaphysis. *Vedda*. SARASIN (1893) mentions the average *SC* of 5 male adult Veddas. The femur is pilastriform, like that of Pĕsĕchĕm-man, therefore I have calculated the sagittal measurements by multiplying the real sagittal diameter in the middle of the diaphysis by the same factor as was done in the Pĕsĕchĕm-man. *Spy I*. FRAIPONT and MAX LOHEST (1887) described two fossil Neanderthal-men. The entire femur of Spy I only was found. In the original article *SC* is not mentioned ("parceque Fraipont a été effrayé des résultats obtenus"). He found namely a capacity of 1562 cc, which was much to high in his opinion, as is mentioned by BOULE (1911—1913) in a footnote in his monograph on the Chapelle-aux-Saints-man. *Chapelle-aux-Saints*. Detailed description of an adult male skeleton of a Neanderthal-man by BOULE (1911—1913). *Combe-Capelle*. Description by KLAATSCH (1909). *SC* was calculated from maximal length, maximal breadth and basion-bregma-height by LEE–PEARSON's formula. KLAATSCH says that the skull is excessively thick, on account of which, after LEE–PEARSON, I have subtracted 50 cc from the amount obtained by their formula. *Grimaldi*. Description by VERNEAU (1906) of the skeleton of an adult woman. *SC* was calculated from maximal length, maximal breadth and basion-bregma-height by LEE–PEARSON's formula. (The second skeleton found at Grimaldi is of a young man, not yet adult and therefore not suitable for this calculation.) *Chancelade*. Description by TESTUT of a skeleton of adult man. *SC* calculated from maximal length, maximal breadth and basion-bregma-height by LEE–PEARSON's formula.

For further information, see my book: Normale en abnormale hersengroei in verbana met de cephalisatieleer.

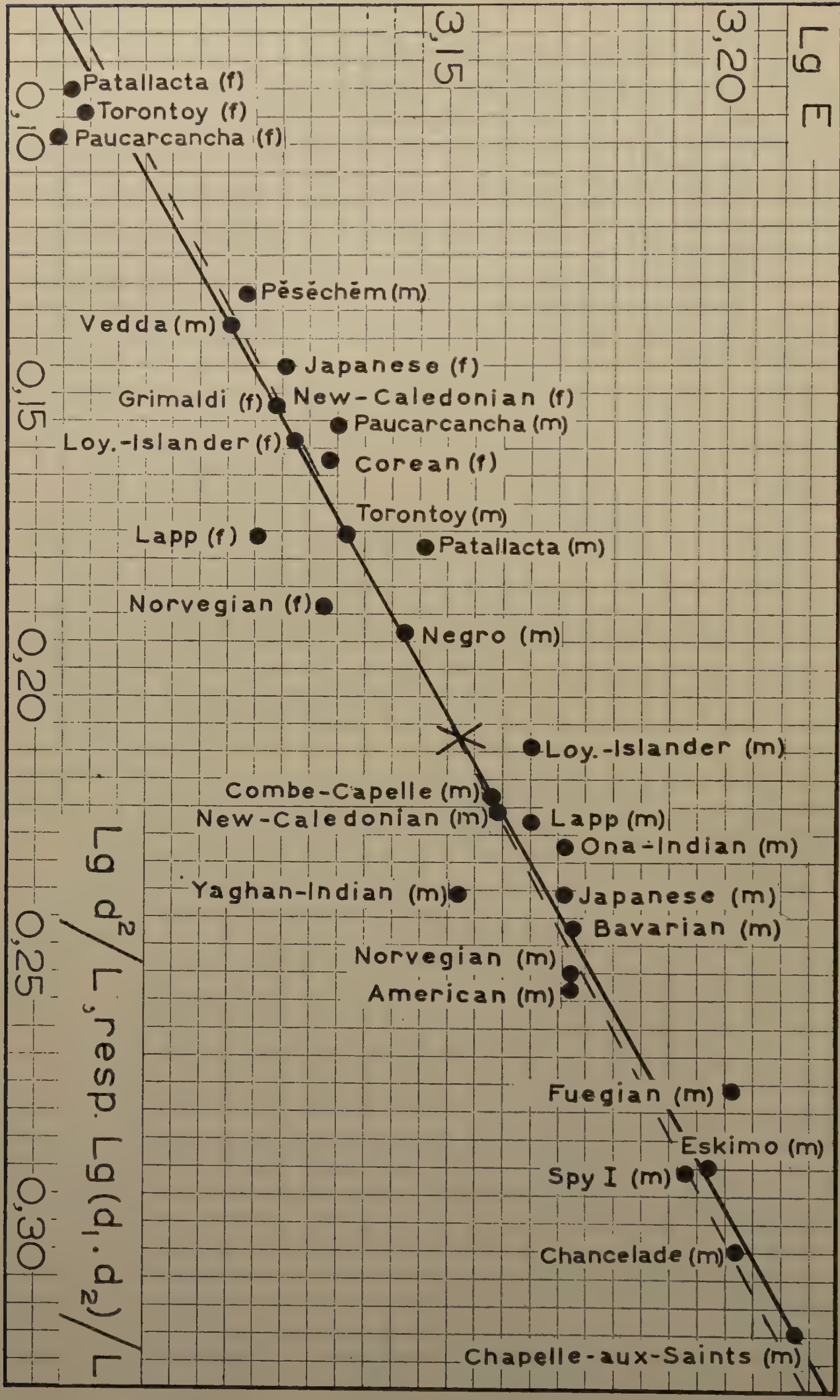
3,15
3,20
Lg E
0,10
0,15
0,20
0,25
0,30
Lg d²/L, resp. Lg(d₁.d₂)/L
Patallacta (f)
Torontoy (f)
Paucarcancha (f)
Pěsěchěm (m)
Vedda (m)
Japanese (f)
Grimaldi (f)
New-Caledonian (f)
Paucarcancha (m)
Loy.-Islander (f)
Corean (f)
Torontoy (m)
Lapp (f)
Patallacta (m)
Norvegian (f)
Negro (m)
Loy.-Islander (m)
Combe-Capelle (m)
New-Caledonian (m)
Lapp (m)
Ona-Indian (m)
Yaghan-Indian (m)
Japanese (m)
Bavarian (m)
Norvegian (m)
American (m)
Fuegian (m)
Eskimo (m)
Spy I (m)
Chancelade (m)
Chapelle-aux-Saints (m)

Fig. 2.

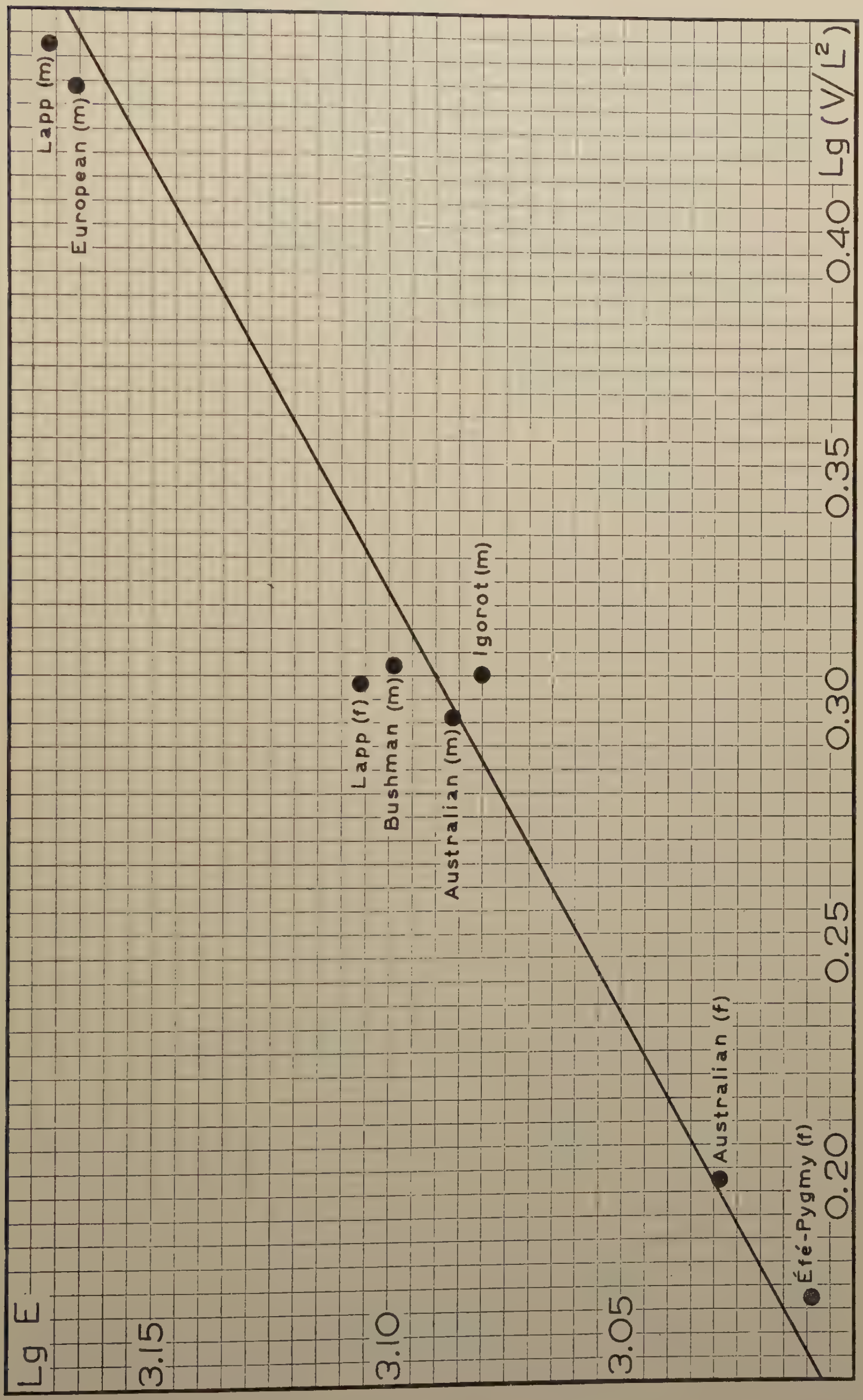
Lg E
3.15
3.10
3.05
Lg (V/L^2)
0.20
0.25
0.30
0.35
0.40
Lapp (m)
European (m)
Lapp (f)
Bushman (m)
Australian (m)
Igorot (m)
Australian (f)
Éfé-Pygmy (f)

Fig. 3.

The total skull capacity is 940 cc according to McGREGOR [1]), who, starting from a cast of the internal cranium, constructed a complete encephalon comparing it with the braincast of higher apes and humans. It would seem that McGREGOR's determination, accurately linking up as it does with the actual data, comes nearest to the actual capacity.

TABLE III [2])

Name	Femur		Skullcapacity (measured) (cc)	Skullcapacity (calculated) (cc)	Error in %
	maximal length (mm)	volume (cc)			
Australian, m.	438	384	1219.2	1220	0.07
Australian, f.	401	256	1065.5	1074	0.80
European, m.	431	505	1466	1446	1.36
Lapp, m.	387	415	1490	1461	1.95
Lapp, f.	364	270	1283	1232	3.98
Igorot, m.	386	305	1200	1234	2.83
Bushman, m.	436	391	1260	1238	1.75
Efé-Pygmy, f.	364	200	1020	1043	1.02

The Pithecanthropus femur found by DUBOIS is complete and in excellent condition. Formula I, however, is not applicable here, owing to a slight pathological growth, an inflammatory exostosis, which, as is clearly seen in DUBOIS' figures, has influenced the middle of the diaphysis (periostitis and local thickening of the diaphysis). With formula I a difference of a few mm in the middle of the diaphysis already causes a considerable difference in the skull capacity calculated. By using the femur *volume* the relation becomes much less sensible to such a local deviation in the diaphysis, for which reason we prefer to apply to Pithecanthropus formula II, which is equivalent to formula I. According to DUBOIS' accurate measurements the maximal length of the Pithecanthropus femur is 445 mm, and the volume, without the exostosis, 463 cc.

When we introduce a factor x into the second term of formula II, substituting the above values for skull capacity and femur measurements, (II) becomes $940 = x.\ 38508\ .\ (463/445^2)^{1/9}$, from which we calculate

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[1]) McGREGOR, Recent studies on the skull and brain of Pithecanthropus erectus. Natural History, Vol. 25, 1925.

[2]) All these data are taken from OTTO HAUGER (Der Gehirnreichtum der Australier und anderer Hominiden, beurteilt nach ihrem Skelet. Anatomische Hefte-Merkel und Bonnet, I Abt. Hft 179, Bd. 59, München u. Wiesbaden, 1921), except those of the Efé-Pygmy, which I owe to the kindness of Prof. Dr. A. J. P. v. D. BROEK, Utrecht (personal communication).

$x = 0.7068 = 0.999/\sqrt{2} = 1/\sqrt{2}$. This calculation shows us that the cephalization stage of Pithecanthropus erectus is $1/\sqrt{2}$ times lower than that of recent and fossil humans.

The data of *Sinanthropus pekinensis* are unfortunately not quite so good as those of Pithecanthropus erectus (one finding place and probably one individual). Yet they allow of an important conclusion.

In the year 1938, Dr. W. C. PEI discovered amond the extensive material recovered from locality 1 at Choukoutien during the excavations of 1936—37, a fragment of a femur (femur *M*) consisting of almost the entire diaphysis. Its length is 312 mm. At the upper end the head and the trochantera are broken of, whereas the entire inferior border of the neck is preserved. The lower end is likewise missing but the upper end of the suprapatellar fossa is still recognizable. Certain parts of the surface are weathered but all the characteristic features are sufficiently preserved to permit a criterion. Its relative completeness makes it possible to determine the general form. Estimated on the basis of the length of the preserved portion of the femur, the total length may have been 400 mm (WEIDENREICH). This femur is considered to belong to a female being on account of the smallness of the two main diameters of the shaft which, in the middle of the diaphysis, are 29.0 and 24.1 respectively. On the same site as femur *M*, according to WEIDENREICH, was found a fragment of a lower jaw with two teeth which also has been attributed to a female individual.

Two years before (1936) three almost complete braincases of Sinanthropus were found in Choukoutien at site *L*. All three skulls represent adult individuals, as proved by the condition of the sutures. According to the size and the general appearance of the skulls (muscle markings, etc.), WEIDENREICH considers skull I and skull III to belong to male individuals and skull II to a female. It is clear that in connection with our problem skull II proves of special interest, as the above mentioned femur also belongs to a female being. According to WEIDENREICH the skull of this female has a capacity of 1050 cc. On the ground of these data we shall now determine the cephalization stage of Sinanthropus, applying formula I. (Formula II cannot be used because of the absence of reliable estimates of the femur volume; on account of the perfect condition of the diaphysis, formula I is eminently suited for our purpose.)

Substituting in (I) 1050 for the skull capacity, 400 for maximal length and 29.9 and 24.1 respectively for the transverse and sagittal measurements in the middle of the diaphysis, and introducing an x-factor into the second term, we get for (I): $1050 = x \,.\, 1084 \,.\, (29.0\ 24.1/400)^{5/9}$, from which we calculate: $x = 0.7105 = 1.008/\sqrt{2} = 1/\sqrt{2}$.

According to these calculations, therefore, the cephalization stages of Pithecanthropus erectus and Sinanthropus pekinensis are the same, and one stage below that of recent and fossil human beings. The agreement in cephalization between the two is striking. On the ground of it we are inclined to consider *Pithecanthropus erectus and Sinanthropus pekinensis as two related types, representing similar stages in evolution.*

The question, whether this relationship is such that the two forms should be considered as belonging to two races of the same species, cannot be answered on the ground of the agreement in cephalization stage. Agreement in cephalization stage does not exclude the possibility of difference in species, and certainly not a difference in race and degree of primitiveness. This is evident from our investigation of recent and fossil human races which differ greatly in primitiveness, while having reached the same stage in cephalization, and also from an investigation of animals, where the same cephalization stage is found in related species (e.g. cat and tiger).

From animal investigation we know that the cephalization stage only indicates the boundary which groups different, nearly related forms together, while separating them from other groups. It is for closer anatomical research to diagnoze, e.g. by studying the fissuration (ARIËNS KAPPERS), the character of this relationship and the greater or lesser primitiveness within this group of related species.

Summary.

The cephalization stages of Pithecanthropus erectus and Sinanthropus pekinensis, measured by the relation existing between skull capacity and femur measurements (formula I and II) are the same; therefore we conclude that these organisms represent a similar stage of evolution.

Anatomy. — *Verdubbeling der fronttanden* (tweede mededeeling). Door TH. E. DE JONGE-COHEN. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of April 27, 1940.)

Toen wij in eene vorige verhandeling in de *Proceedings* opnieuw het probleem der tandverdubbeling ter sprake brachten (1), ontleenden wij onze voorbeelden aan die gevallen, wier vormgenese op transversale klieving van den oorspronkelijken tandaanleg berustte (*schizodontie*) [1].

In aansluiting nu aan deze mededeeling willen wij thans ten tweede male enkele beschouwingen aan hetzelfde onderwerp wijden en een geheel anderen, veel zeldzamer vorm van tandverdubbeling bespreken: het is die, welke als gevolg van longitudinale concrescentie van twee — oorspronkelijk gescheiden — tandkiemen tot stand komt en welken wij uit dien hoofde reeds eerder als *synodontie* bestempelden (2) [2].

Vooraf enkele bijzonderheden over onze gevallen.

* * *

Aantal.

Het totale aantal bedroeg 14, verdeeld over 1 bovenkaak en 11 onderkaken: in 2 daarvan droeg de anomalie een bilateraal-symmetrisch karakter (afb. 5 en 13) en met nadruk onderstrepen wij de waarlijk unieke zeldzaamheid dezer symmetrie!

Verdeeling over de verschillende gebitselementen.

Naast versmelting van medialen met lateralen snijtand kennen wij als tweede mogelijkheid coalescentie van dezen laatsten en hocktand. Zoo blijkt de anomalie de mediaanlijn derhalve niet te overschrijden: synodontie der beide mediale incisivi komt óók in ons materiaal niet voor.

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[1] BOLK (2) had deze reeds vroeger in zijne *Odontologische Studiën* als schizogene variaties beschreven, in deze benaming tevens de formale genese der anomalie tot uitdrukking brengende.

Teneinde echter haar karakter als afwijking *sui generis* scherper tegenover de in deze mededeeling te bespreken dubbeltanden te accentueeren, willen wij haar als *schizodontie* aanduiden.

[2] Aan de omschrijving „*synodontie*" is, gelijk wij destijds reeds betoogden, niet slechts boven de gebruikelijke termini technici het voordeel eener in hare beknoptheid duidelijke omschrijving verbonden, doch ligt evenzeer de gedachte ten grondslag, onze nomenclatuur zoo volkomen mogelijk met de reeds bestaande terminologie in overeenstemming te brengen; c.f. anodontie, syndactylie, sympodie.

Verdeeling over boven- en onderkaak.

Met uitzondering van één specimen in de bovenkaak kwamen al onze gevallen in de onderkaak tot ontwikkeling. Ook onder dit opzicht stemmen onze objecten vrijwel volkomen overeen met de in de literatuur neergelegde vondsten, welke immers eveneens bijkans alle op de onderkaak betrekking hebben.

Geenerlei gevallen daarentegen bij de blijvende boventanden ³). De verklaring daarvan behoeven wij niet verre te zoeken. Versmelting toch is slechts mogelijk in het allervroegste stadium der tandontwikkeling en dan nog alleen tusschen die elementen, welke in hun tijdstip van aanleg en verkalking niet al te sterk divergeeren. Aldus is in het melkgebit bij alle fronttanden het geval: óók tusschen lateralen snijtand en cuspidatus bestaat ten deze slechts een gering tijdsinterval. Hetzelfde geldt m.m. voor de blijvende serie in de onderkaak: in de bovenkaak daarentegen legt de sterk geretardeerde ontwikkeling van den lateralen incisivus versmelting van dezen met zijne beide synergeten onoverkomelijke moeilijkheden in den weg.

Verdeeling over melkdentitie en blijvende serie.

Terwijl 10 gevallen het melkgebit betroffen, telden wij de anomalie slechts 4 malen in de blijvende rij.

Stellen deze cijfers — in volkomen overeenstemming met de bevindingen van vroegere onderzoekers — het dominante karakter der melkdentitie ten deze in het licht, nog duidelijker komt dit naar voren, wanneer wij, met uitsluiting der gipsafgietsels, onze telling tot de in de mandibulæ aanwezige objecten bepalen. Deze toch zijn alle van de sceletresten afkomstig van twee groote kerkhoven en komen derhalve de verhoudingen *durante vita* het dichtst nabij! Welnu: op een totaal aantal van ongeveer 36.000 schedels vonden wij slechts 1 casus van synodontie in het blijvende gebit tegen 3 in de melkdentitie (v.d. afb. 2, 8, 9 en 3).

Doch ook deze cijfers vormen geen zuiveren maatstaf: immers dienen wij niet uit het oog te verliezen, dat het aantal opgegraven volwassen crania dat der kinderschedeltjes vele malen overtreft. Nemen wij bovendien echter in aanmerking, dat juist van deze laatste een niet zoo heel gering percentage — en dan vermoedelijk juist de voor ons onderzoek meest belangwekkende — ingevolge eene vrij langdurige inhumatieperiode verloren gegaan zijn, dan is de conclusie gewettigd, dat het verhoudingscijfer voor het melkgebit in werkelijkheid nog belangrijk hooger ligt.

Anderzijds echter vormt ons materiaal in zijn uitgesproken voorliefde voor de melktanden een scherp contrast met de gevallen van schizodontie. De morphologische variaties der fronttanden toch maken zich alle zonder

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³) Gevallen van secundaire vergroeiing nà beëindigde odontogenese (ingevolge hypertrophie van het cæmentum — apicale parodontitis) vallen buiten het begrip synodontie en blijven in deze mededeeling derhalve *eo ipso* buiten beschouwing.

uitzondering bij voorkeur in de bovenkaak en dan met name in het blijvend gebit kenbaar: in de onderkaak is hunne frequentie aanzienlijk geringer! Identieke verhoudingen vinden wij m.m. in de melkdentitie terug met dien verstande, dat hier de overeenkomstige anomalieën nòg veel zeldzamer zijn dan in het blijvend gebit — m.a.w.: *het laagste percentage bij de melkelementen in de onderkaak*. Ook de schizogene varianten maken op dezen regel geen uitzondering.

Juist omgekeerd nu blijkt de stand van zaken bij synodontie: deze manifesteert zich nagenoeg uitsluitend in de onderkaak en dan bij voorkeur bij de melkelementen!

Dit onderscheid is te opmerkelijk om niet een oogenblik onze aandacht te vragen. Want het accentueert wel op bijzonder fraaie wijze het genetisch verschillend karakter van beide afwijkingen: immers dienen wij geenszins uit het oog te verliezen, dat synodontie — in tegenstelling met schizodontie — niet onder de eigenlijke morphologische gebitsvariaties gerangschikt mag worden. De laatste toch berusten alle op de activeering van latente ontwikkelingspotenties, synodontie daarentegen vormt het substraat eener ontwikkelingsstoornis.

⁂

Het clinisch beeld der synodontie vertoont in zijn wisselend adspect sprekende gelijkenis met dat der schizodontie: deze blijkt niet slechts ectoscopisch aan de kroon waarneembaar, doch wordt in gelijke mate door het RÖNTGENbeeld der wortelformatie bevestigd.

Begrijpelijk derhalve, dat wij in onze vorige mededeeling over dit onderwerp met nadruk betoogden: „dat ook synodontie — eveneens bij voorkeur bij de voortanden — aanleiding kan geven tot de vorming van z.g. tweelingstanden, welke morphotisch alleszins, genetisch geenszins identiek zijn met de gevallen van echte tandverdubbeling" [4]).

Moge deze isomorphie in het licht van onze bovengegeven beschouwingen paradoxaal schijnen, zoodra wij ons rekenschap geven van de ontstaanswijze der dubbeltanden, is het duidelijk, dat wij — ondanks hun verschillend genetisch karakter — èn bij schizodontie èn bij synodontie in feite met eenzelfden ontwikkelingsmodus te doen hebben. Bij schizodontie: longitudinale differentiatie van den oorspronkelijken tandaanleg, daarnaast transversale insnoering met als eindstadium twee *zelfstandige* gebitselementen.

Juist omgekeerd nu verloopt dit proces bij synodontie: hier twee oorspronkelijk gescheiden tandkiemen, die hunne eigene individualiteit — waaraan slechts eene transversale insnoering de herinnering gefixeerd houdt (afb. 8 en 9) — niet wisten te handhaven: longitudinale concentratie en als eindstadium *één* ietwat verbreede tand (afb. 11).

Maar aldus levert in ons dentaal systeem de formale genese der dubbeltanden tevens het marquante voorbeeld van een ontwikkelingsgang, die in twee — aan elkander tegengestelde — richtingen verloopen kan.

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[4]) l.c. pag. 503—504.

En wanneer wij in aansluiting aan het bovenstaande tenslotte Uwe aandacht vragen voor den ook in de literatuur een enkele maal beschreven *„drielingstand"*, dan is het duidelijk, dat wij hier te doen hebben met eene wel uitzonderlijk zeldzame combinatie van synodontie met schizodontie: *van onze beide gevallen* (afb. 6 en 7) *manifesteerde één zich in de bovenkaak.*

* * *

Nu zijn wij ons terdege bewust, met de door ons gegeven voorstelling van zaken een van de heerschende opvattingen ten eenenmale afwijkend standpunt in te nemen. Want ofschoon GUILFORD (4) en WEDL (5) reeds nadrukkelijk onderscheid maakten tusschen echte tweelingstanden en tandversmeltingen, houden met name de Duitsche onderzoekers — van welke wij in het bijzonder PECKERT (6) vermelden, omdat deze en zijne leerlingen de literatuur met een aantal belangwekkende casuïstische bijdragen verrijken — tot op den huidigen dag nagenoeg algemeen aan de door BUSCH (7) gegeven indeeling vast [5]).

Deze beschouwt niet slechts de echte synodontiegevallen als coalescentieverschijnsel doch ziet óók in de door ons als tweelingstanden beschreven gevallen van schizodontie het substraat eener versmelting — te weten tusschen een normaal en een overtollig element.

In korte samenvatting zouden wij zijne zienswijze aldus kunnen formuleeren: tandverdubbeling berust op samensmelting van twee normale elementen dan wel van een normalen met een overtolligen tand. Daartegenover ligt aan onze zienswijze de gedachtengang ten grondslag, dat wij bij dubbeltanden terdege hebben te onderscheiden tusschen echte tweelingstanden (*schizodontie*) en die vormen, welke door fusie van twee normale gebitselementen (*synodontie*) tot stand komen.

De boven besproken vormovereenkomst verdient echter nog in ander verband onze aandacht. De vraag is: hoe kunnen wij de differentieele diagnose tusschen beide dubbeltandformaties stellen? Vooropstellende, dat moeilijkheden ten deze alleen dan te verwachten zijn, wanneer — met uitsluiting derhalve der cuspidati — de dubbeltand zich binnen het gebied der snijtanden manifesteert, kan bij de beantwoording der vraag de bovenkaak nochtans gevoegelijk buiten beschouwing blijven. In de eerste plaats, wijl — gelijk wij reeds eerder zagen — synodontie hier tot de allergrootste uitzonderingen behoort: slechts zéér enkele gevallen zijn beschreven [6]) en

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[5]) Voor HERBST (8) maken wij nadrukkelijk eene uitzondering.

[6]) Een buitengewoon belangwekkende mededeeling vonden wij bij LEIGH (9). Deze toch verrijkte de literatuur met de beschrijving van een geval van bilaterale synodontie in de bovenkaak: versmelting van de drie melkfronttanden rechts en daarnaast van medialen en lateralen incisivus links achten wij een zóó uitzonderlijk zeldzaam phænomeen, dat wij de vermelding ervan niet achterwege willen laten.

zelfs deze zouden wij niet gaarne alle zonder meer voor onze rekening durven nemen. Daarnaast verlieze men niet uit het oog, dat vorm en grootte der beide incisivi in de bovenkaak gemeenlijk véél sterker divergeeren dan in de onderkaak. Maar tenslotte: reeds vroeger hebben wij er den nadruk op gelegd, dat bij den echten tweelingstand de sulcus, welke zijne beide segmenten onderling afgrenst, vrijwel altijd een smallere mediale zône van een ietwat breedere distale scheidt. Welnu: juist omgekeerde verhoudingen zouden wij in de bovenkaak bij vergroeiing van medialen met lateralen snijtand mogen verwachten!

In de onderkaak daarentegen kan het beeld der synodontie in melkdentitie en blijvend gebit zóó sprekende gelijkenis vertoonen met den echten tweelingstand, dat wij de verleiding niet weerstaan kunnen, twee voorbeelden van dezen laatsten weer te geven (afb. 14 en 15).

De isomorphie der beide dubbeltandformaties blijkt hier zóó volkomen, dat slechts hare relatie tot de omgeving haar morphologisch karakter bepalen kan. Daarbij kan „*Symmetrievergleich*" van niet te onderschatten beteekenis blijken, maar beslissend voor elken casus is het aantal der tanden.

Laten wij die gevallen buiten beschouwing, bij welke de splitsingstendenz haar eindstadium bereikt heeft en de tweelingstand tot twee tanden uitgegroeid is, dan kunnen wij vaststellen, dat bij schizodontie het oorspronkelijke aantal bewaard blijft, terwijl dit bij synodontie uitteraard met één verminderd is — overigens normale gebitsverhoudingen vooropgesteld natuurlijk.

Waar de diagnose derhalve in hoofdzaak *ex juvantibus* gesteld moet worden, zal het practisch vrijwel nooit mogelijk blijken, geïsoleerde gevallen van snijtandsverdubbeling nader te determineeren.

⁂

Tenslotte vraagt ook de causale genese een oogenblik onze aandacht.

Terwijl de ontplooiing eener sluimerende ontwikkelingspotentie het beeld der schizodontie morphologisch het karakter eener progressieve vormdifferentiatie verleent, dragen de gevallen van synodontie alle den stempel eener morphotisch regressieve ontwikkelingstendenz.

Voor hunne totstandkoming zouden wij mèt STURM (10) ruimtegebrek aansprakelijk willen stellen, wel te verstaan in dien zin, dat te geringe afstand tusschen twee tandkiemen zelfstandige ontwikkeling van *beide* in den weg staat (afb. 1).

Aldus een vitium primæ formationis, vormt synodontie een opmerkelijke bijdrage tot de pathologie der ontwikkeling!

SAMENVATTING.

Tandverdubbeling ingevolge synodontie der fronttanden kennen wij bij beide dentities. Bij voorkeur echter manifesteert zij zich bij de ondertanden van het melkgebit. Naast coalescentie van medialen met lateralen snijtand kennen wij bovendien versmelting van lateralen incisivus met cuspidatus: de mediaanlijn daarentegen schijnt de anomalie niet te overschrijden!

Hare vormgenese berust op longitudinale concrescentie van twee oorspronkelijk gescheiden tandkiemen en verloopt aldus geheel tegengesteld aan die der schizodontie.

ZUSAMMENFASSUNG.

Zahnverdoppelung infolge Synodontie der vorderen Zähne ist bei beiden Dentitionen bekannt. Vorzugsweise tritt sie bei den unteren Zähnen des Milchgebisses in Erscheinung. Neben der Coalescenz des medialen mit dem lateralen Schneidezahn ist uns ausserdem die Verschmelzung des lateralen Incisivus mit dem Cuspidatus bekannt: die Anomalie jedoch überschreitet anscheinend die Medianlinie nicht.

Ihr Formgenese beruht auf längswändiger Verschmelzung zweier ursprünglich geschiedenen Zahnkeimen und verläuft somit ganz entgegengesetzt an der der Schizodontie.

RESUMÉ.

Redoublement des dents à cause de synodontie des incisives est connu des deux dentitions. Il se manifeste de préférence aux dents de lait inférieures. A part de la coalition de la médiale avec l'incisive latérale, nous connaissons également la fusion de l'incisive latérale avec la canine; l'anomalie par contre ne paraît pas dépasser la ligne médiane.

Sa génèse est fondée sur la concrescence longitudinale de deux germes de dents séparées à l'origine et se fait donc tout à fait en sens contraire de celle de la schizodontie.

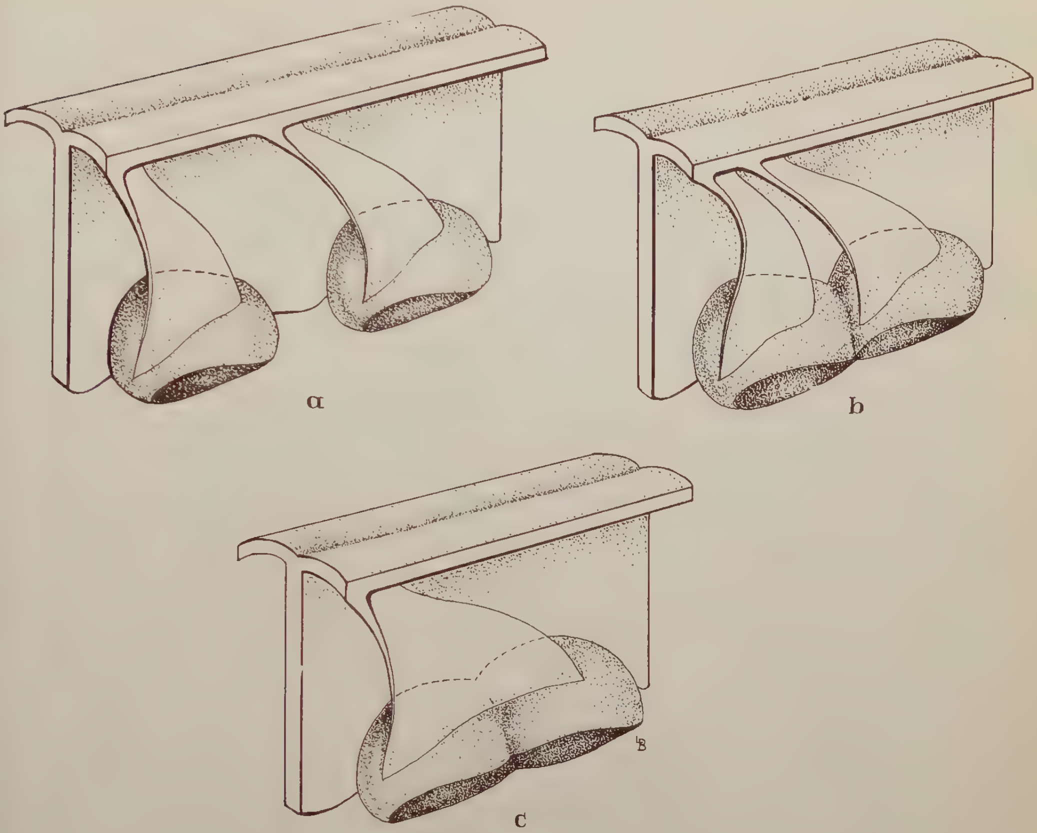
SUMMARY.

Gemination of the teeth in accordance with synodontia of the frontteeth is wellknown in both dentitions. By preference however gemination appears in the milkteeth of the lower jaw.

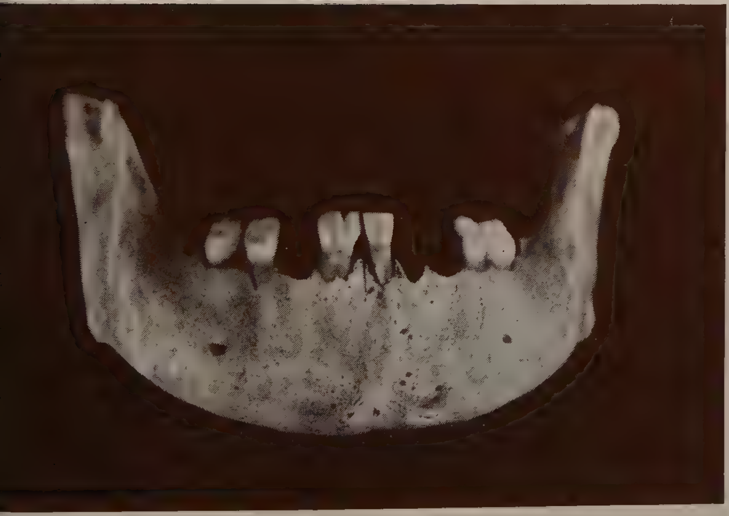
Beside coalescence of the medial and lateral incisor we know moreover the fusion of the lateral incisor with cuspidate: the anomaly on the other hand seems not to cross the medianline.

Its genesis is based on longitudinal concrescence of two originally separated toothgerms, and takes place in a sense opposite to that of schizodontia.

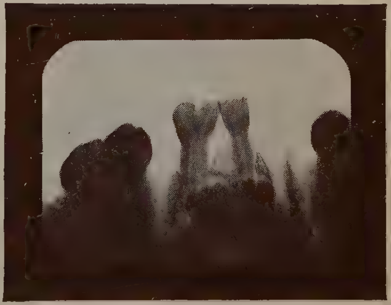
Anatomisch-embryologisch Instituut der Universiteit
van Amsterdam. Mauritskade 61, O..



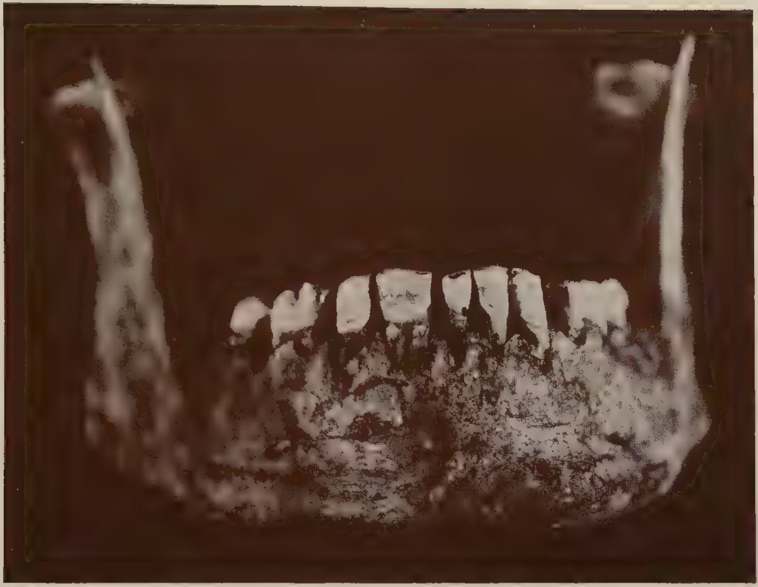
Afb. 1.



Afb. 2*a*.

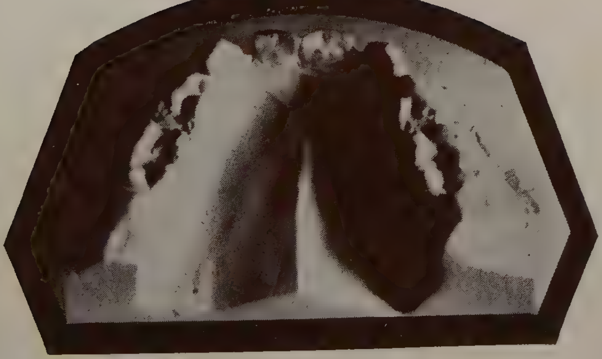


Afb. 2*b*.

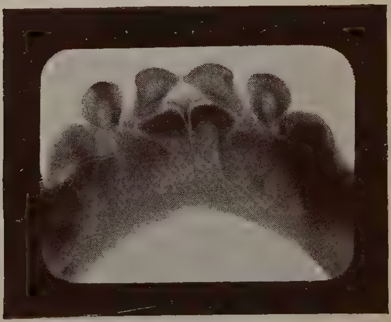


Afb. 3.

Afb. 4.

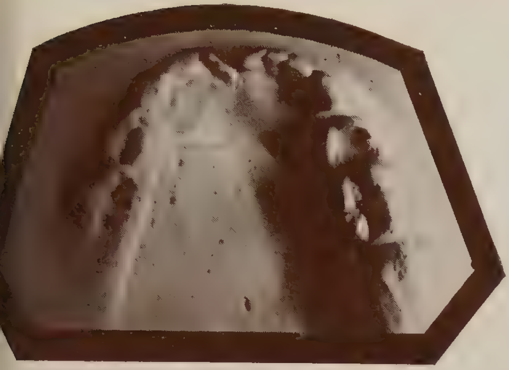


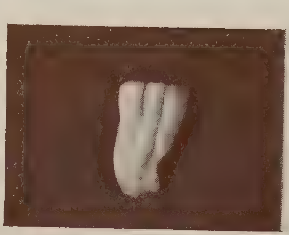
Afb. 5*a*.



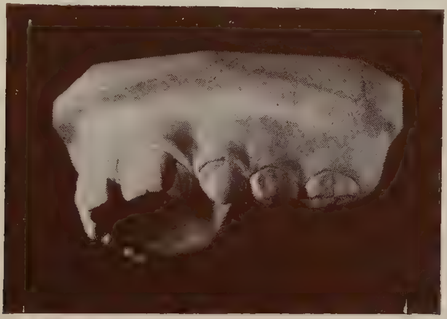
Afb. 5*b*.

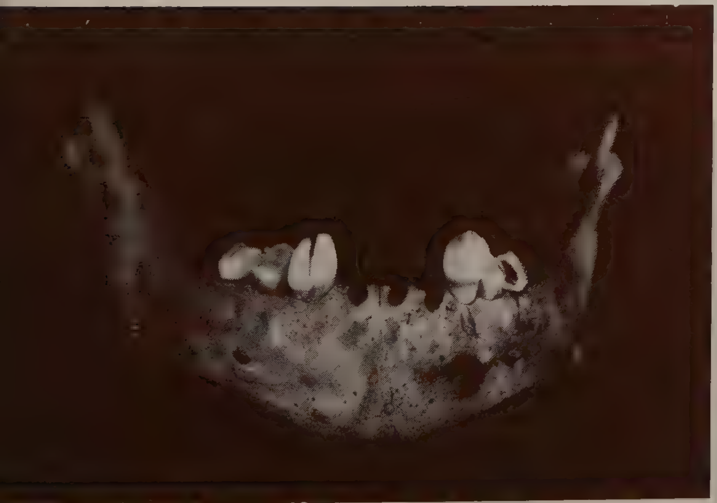
TH. E. DE JONGE-COHEN: VERDUBBELING DER FRONTTANDEN. II

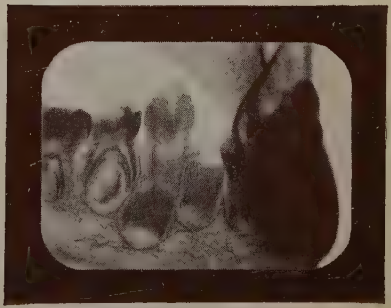

Afb. 6*a*.


Afb. 6*b*.

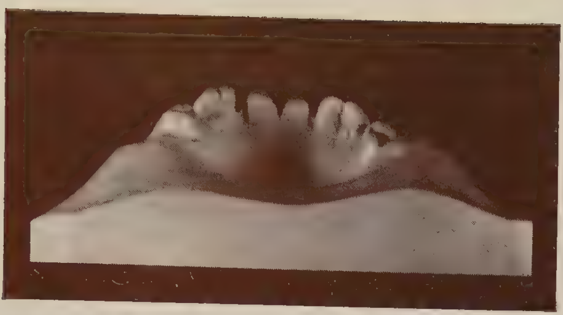
Afb. 6*c*.


Afb. 7.

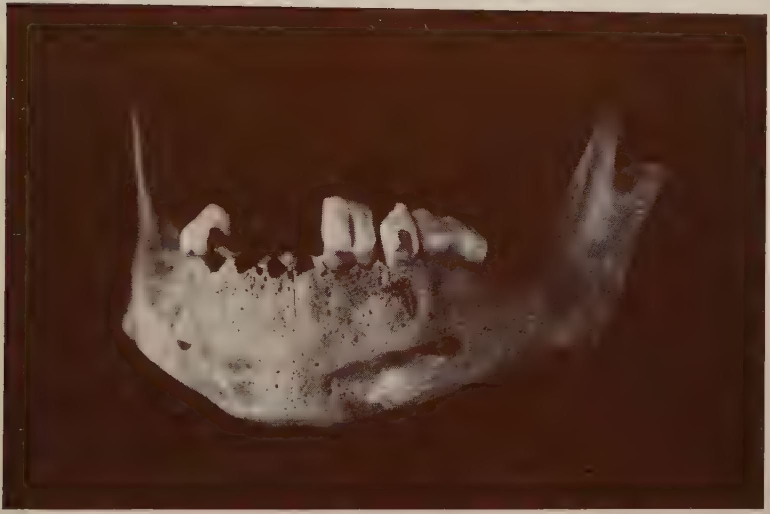

Afb. 8*a*.


Afb. 8*b*.

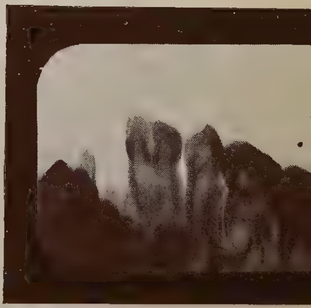
Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.



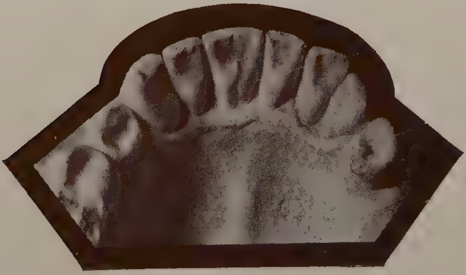
Afb. 13 (van oraal gezien).



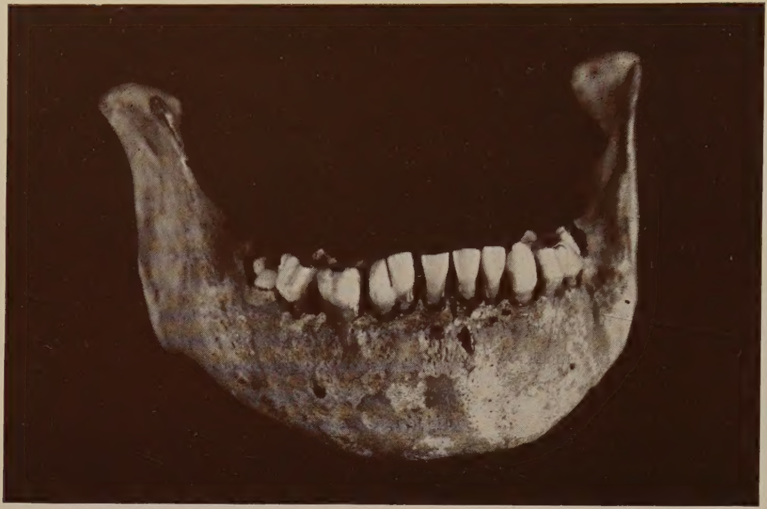
Afb. 14*a*.



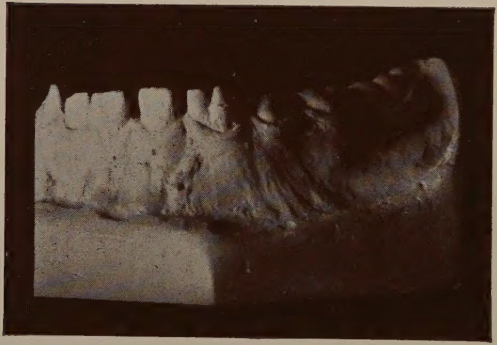
Afb. 14*b*.



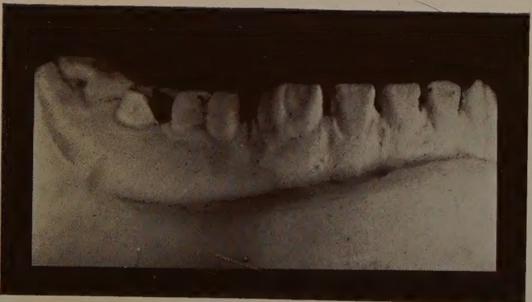
Afb. 15.



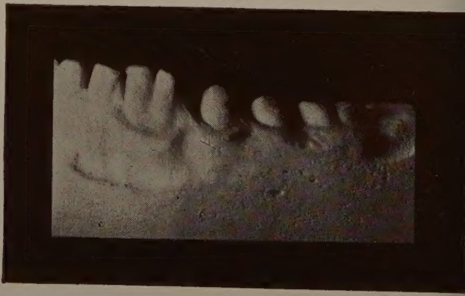
Afb. 9.



Afb. 10.



Afb. 11.



Afb. 12.

TOELICHTING DER AFBEELDINGEN.

Afb. 1*a*, *b* en *c* : Drie opeenvolgende phases in de vormgenese der synodontie.

Synodontie van medialen en lateralen incisivus.

Afb. 2*a* en *b* : melkdentitie,
afb. 3 en 4 : blijvend gebit,
afb. 5*a* en *b* : geval van bilateraal-symmetrische synodontie der beide snijtanden,
afb. 6*a*, *b*, *c* en 7 : synodontie der beide elementen in onder- en bovenkaak, gepaard gaande met schizodontie: dens trigeminus!

Synodontie van lateralen incisivus en cuspidatus.

Afb. 8*a* en *b*, 9 en 10: melkdentitie,
afb. 11 en 12 : blijvende serie,
afb. 13 [7]) : geval van bilateraal-symmetrische synodontie.

Schizodontie.

Afb. 14*a* en *b* : lacteale serie (laterale incisivus),
afb. 15 : permanent gebit (mediale incisivus).

BIBLIOGRAPHIE.

1. DE JONGE-COHEN, TH. E., Verdopplung der Frontzähne. *Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam,* **41**, Afl. 2 (1938).
   Tijdschrift voor Tandheelkunde, Jaargang XLV, Afl. 6 (1938).
2. BOLK, L., Odontologische Studiën. II Die Morphogenie der Primatenzähne. Eine weitere Begründung und Ausarbeitung der Dimertheorie, Jena (1914).
3. DE JONGE-COHEN, TH. E., Incisivi geminati. *Tijdschrift voor Tandheelkunde,* Jaargang XXXV, afl. 11 (1928).
4. GUILFORD, S. U., American System of Dental Surgery, Band III (1887).
5. WEDL, C., Pathologie der Zähne mit besonderer Rücksicht auf Anatomie und Physiologie, Leipzig (1870).
6. PECKERT, H., Die Morphologie der Missbildungen des Menschen und der Tiere. Die Missbildungen des Gebisses, Jena (1911).
   Zwillingsbildung. Handwörterbuch der gesamten Zahnheilkunde.
7. BUSCH, Ueber Verschmelzung und Verwachsung der Zähne des Milchgebiszes und des bleibenden Gebisses. *Deutsche Monatschrift für Zahnheilkunde,* Jaargang XV, Afl. 11 (1897).
8. HERBST, E., Doppelbildungen der Zähne; eine vergleichende Studie. *Deutsche zahnärztliche Wochenschrift,* Jaargang XXXVII, Afl. 47 (1937).
9. LEIGH, W. J., Two cases of gemination. *The Britisch Dental Journal,* Band XLIII, Afl. 19 (1926).
10. STURM, F., Seltener Fall von Verschmelzung zweier Milchfrontzähne. *Deutsche zahnärtzliche Wochenschrift,* Jaargang XL, Afl. 38 (1937).
    Zahnanomalien. Ut supra, Jaargang XLI, Afl. 23 (1938).

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[7]) Wij danken dezen casus aan de vriendelijke bereidwilligheid van Dr. STURM te *Gerstungen,* die hem zelf elders reeds beschreef (10).